Weakly asymptotically hyperbolic solutions to the Einstein constraint equations

Paul T. Allen
Lewis & Clark College

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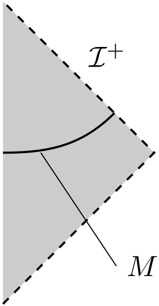
- joint with James Isenberg, John M. Lee, Iva Stavrov Allen
- arXiv: `1506.03399, 1506.06090`; to appear in CAG, CQG

Motivation

Minkowski spacetime

- Conformal structure:



- Foliate by hyperboloids M:

$$g = \breve{g}, \quad K = -\breve{g}$$

Asymptotically flat/simple spacetime

- Conformal structure asymptotically Minkowskian
- CMC hyperboloidal slice M:
  - At conformal boundary

$$\mathrm{Riem}[g] \to -\mathrm{id}, \quad K \to -g$$

  - $\operatorname{tr}_g K = -3$
  - Shear-free condition
    $\to$ necessary for regular $\mathcal{I}^+$

Goal: Construct CMC hyperboloidal data

Data (g, K) on M satisfies

- (CMC) extrinsic curvature $K = -g + \Sigma$ with $\mathrm{tr}_g \Sigma = 0$
- (H) metric g is "asymptotically hyperbolic" and Σ decays
- Constraint equations are satisfied

$$\mathrm{R}[g] - |\Sigma|_g^2 + 6 = 0 \qquad \mathrm{div}_g \Sigma = 0$$

- (SF) "Shear-free condition" holds

First work:

- [ACF] Andersson, Chruściel, Friedrich: $\Sigma \equiv 0$, regularity
- [AC1], [AC2] Andersson, Chruściel: $\Sigma \not\equiv 0$, (SF) necessary

Main issues:

- regularity at conformal boundary
- ensuring shear-free condition

Analysis in asymptotically hyperbolic setting

- $M = \mathrm{int}(\overline{M})$
- defining function $\rho \in C^\infty(\overline{M})$
- weighted spaces $C^{k,\alpha}_\delta(M) = \rho^\delta C^{k,\alpha}(M)$

Metric g is C^k asymptotically hyperbolic if

- conformally compact: $\overline{g} := \rho^2 g \in C^k(\overline{M})$
- $|d\rho|_{\overline{g}} \to 1$ at $\partial M \quad \leftrightarrow \quad \mathrm{Riem}[g] \to -\,\mathrm{id}$

Scalar equation $\Delta u = f$

- $\Delta u = \rho\partial_\rho(\rho\partial_\rho - 2)u + \rho^2 \not\Delta u + \ldots$
- $\Delta \colon C^{k+2,\alpha}_\delta \to C^{k,\alpha}_\delta$ Fredholm/invertible for $0 < \delta < 2$
- resonances at ρ^0 and $\rho^2 \rightsquigarrow$ log terms in formal expansions

Boundary regularity for Yamabe problem

[ACF] study $\mathrm{R}[\phi^4 g] = -6$ with $\rho^2 g \in C^\infty(\overline{M})$ using

$$\Delta_g \phi = \frac{1}{8}\,\mathrm{R}[g]\phi + \frac{3}{4}\phi^5$$

With $\phi = 1 + u$ this becomes

$$(\rho\partial_\rho + 1)(\rho\partial_\rho - 3)u + \cdots = \underbrace{\mathrm{R}[g] + 6}_{\mathcal{O}(\rho)} + Q(u).$$

Lessons:

- solution $u \in C_1^\infty(M)$

  $\rightsquigarrow$ decay of scalar curvature determines weight
- resonances at ρ^{-1} and ρ^3

  $\rightsquigarrow$ possible that $u \in C^2(\overline{M})$, but not $C^3(\overline{M})$

  $\rightsquigarrow$ obstruction to smoothness is "shear tensor"

Boundary regularity for CMC constraints

[AC1],[AC2] extend [ACF] analysis to CMC constraints:

- Use conformal method in strongly AH setting ($\rho^2 g$ smooth, $C^{k,\alpha}$ with $k \geq 2$)
- "Generic" solutions have polyhomogeneous formal expansions $\sum_{j,k} a_{j,k}\, \rho^j (\log \rho)^k$, are not smooth on $\overline{M}$
- Smoothness on $\overline{M}$ requires shear-free condition:

$$\rho\,(\text{traceless } K) = \text{traceless}\,\text{Hess}_{\overline{g}}(\rho) \quad \text{along } \partial M \qquad \text{(SF)}$$

- (SF) is necessary for spacetime develoment with regular $\mathcal{I}^+$
- "Most" solutions constructed in [AC1] do not satisfy (SF)

Needed:

- a regularity class where the conformal method closes
- a way to build shear-free condition in to conformal method

CMC conformal method

- Fix a metric g and tensor traceless tensor μ. Seek initial data of the form

$$\phi^4 g, \quad \Sigma = \phi^{-2}(\mu + \mathcal{D}_g W); \qquad \mathcal{D}_g W = \text{ tracefree } \mathcal{L}_W g$$

- Constraints satisfied in ϕ and W satisfy the elliptic system

$$\mathcal{D}_g^* \mathcal{D}_g W = -\operatorname{div}_g \mu$$
$$\Delta_g \phi = \frac{1}{8} \mathrm{R}[g]\phi - \frac{1}{8}|\mu + \mathcal{D}_g W|_g^2 \phi^{-7} + \frac{3}{4}\phi^5$$

- Want a regularity class of metrics that is
  - strong enough for elliptic theory and defining (SF)
  - weak enough that $g \mapsto \phi^4 \lambda$ closes
- Want a conformally invariant expression of (SF)

Regularity classes: previous work

Andersson & Chruściel [AC1]

- "...introduce a large number of function spaces, probably more than seems reasonable at first sight"
- detailed description of boundary regularity

Gicquaud & Sakovich [GS]

- local Sobolev spaces

Intermediate Hölder spaces

Fix metric $\overline{h} \in C^\infty(\overline{M})$; associated AH metric $h = \rho^{-2}\overline{h}$ on M

- For covariant 2-tensors: $|u_{..}|_{\overline{h}} = \rho^{-2}|u_{..}|_h$ thus

$$u \in L^\infty(\overline{M}) \quad \leftrightarrow \quad u \in L^\infty_2(M) = \rho^2\, L^\infty(M)$$

- $u \in C^k(\overline{M})$ if $\mathcal{L}_{\overline{X}_1} \dots \mathcal{L}_{\overline{X}_l} u \in C^0(\overline{M})$ for $l \leq k$, $\quad |\overline{X}_j|_{\overline{h}} \lesssim 1$
- $u \in C^k(M)$ if $\mathcal{L}_{X_1} \dots \mathcal{L}_{X_l} u \in C^0(M)$ for $l \leq k$, $\quad |X_j|_h \lesssim 1$

Hybrid spaces: 2-tensor $u \in \mathscr{C}^{k,\alpha;m}(M)$ if

$$\underbrace{\mathcal{L}_{X_1} \dots \mathcal{L}_{X_p}}_{p \leq k-m} \underbrace{\mathcal{L}_{\overline{X}_1} \dots \mathcal{L}_{\overline{X}_q}}_{q \leq m} u \in C^{0,\alpha}_2(M)$$

$\mathscr{C}^{k,\alpha;m}(M)$ is intermediate to $C^{k,\alpha}_2(M)$ and $C^{k,\alpha}(\overline{M})$

Note: These are not *the V_b spaces of Melrose–Mazzeo*

Weakly asymptotically hyperbolic metrics

Mantra

- "high" interior regularity
- "low" boundary regularity

Defn: g is *weakly asymptotically hyperbolic* of class $\mathscr{C}^{k,\alpha;1}$ if

- $\overline{g} = \rho^2 g \in \mathscr{C}^{k,\alpha;1}(M)$
  - $\overline{g} \in C_2^{k,\alpha}(M) \quad \longrightarrow \quad \overline{g} \in L^\infty(\overline{M})$
  - $\mathcal{L}_{\overline{X}}\overline{g} \in C_2^{k-1,\alpha}(M) \quad \longrightarrow \quad \overline{g} \in W^{1,\infty}(\overline{M}) \subset C^{0,1}(\overline{M})$
- $|d\rho|_{\overline{g}} = 1$ along ∂M
  - $\mathrm{Riem}[g] \to -\mathrm{id}$

A—, Isenberg, Lee, Stavrov Allen

- Such regularity is sufficient for elliptic theory

Elliptic theory

Previous work in asymptotically hyperbolic setting

- edge calculus [Mazzeo]
- geometric approach: [Andersson], [Lee]
- all require at least C^2 conformal compactification

We adapt the geometric approach of [Lee]

- Let g weakly AH of class $\mathscr{C}^{k,\alpha;1}$
- P is a second-order geometric elliptic operator arising from g then

$$P\colon C^{k,\alpha}_{\delta}(M) \to C^{k-2,\alpha}_{\delta}(M)$$
$$P\colon W^{k,p}_{\delta}(M) \to W^{k-2,p}_{\delta}(M)$$

are Fredholm of index zero for all δ in "Fredholm range"

Elliptic theory, continued

Key ideas of proof

- Fredholm range determined by model operator $\breve{P}$, corresponding to hyperbolic metric.
- Interior $C^{k,\alpha}$ regularity $\longrightarrow$ elliptic estimates, etc.
- Boundary $C^{0,1}$ regularity $\longrightarrow$ parametrix construction

Application to constraint equations

- We can solve the elliptic system in the conformal method
- Method closes: If g is weakly AH, then $\phi^4 g$ also weakly AH.

The shear-free condition

We want to enforce

$$\rho\Sigma = \operatorname{traceless}\operatorname{Hess}_{\overline{g}}\rho \quad \text{along } \partial M. \tag{SF}$$

Two issues:

- g weakly AH of class $\mathscr{C}^{k,\alpha;1}$ only implies $\overline{g} \in C^{0,1}(\overline{M})$
- Want to enforce (SF) in conformally invariant manner

Define g weakly AH of class $\mathscr{C}^{k,\alpha;2}$ if $\overline{g} \in \mathscr{C}^{k,\alpha;2}(M)$

- $\overline{g} \in W^{2,\infty}(\overline{M}) \subset C^{1,1}(\overline{M}) \quad \rightsquigarrow$ (SF) condition defined
- solution map $g \mapsto \phi^4 g$ closes (with some work; scalar curvature)

Conformally-invariant shear-free condition

Define traceless tensor

$$\mathcal{H}_{\overline{g}}(\rho) = |d\rho|^6_{\overline{g}} \underbrace{\mathcal{D}_{\overline{g}}(|d\rho|^{-2}_{\overline{g}} \operatorname{grad}_{\overline{g}} \rho)} + \tfrac{1}{2}|d\rho|_{\overline{g}} \underbrace{\operatorname{div}_{\overline{g}}(|d\rho|_{\overline{g}} \operatorname{grad}_{\overline{g}} \rho)} (d\rho \otimes d\rho - \tfrac{1}{3}|d\rho|^2_{\overline{g}} \overline{g})$$

- Conformal covariance $\mathcal{H}_{\theta\overline{g}}(\rho) = \theta^{-2}\mathcal{H}_{\overline{g}}(\rho)$
- If g is AH of class $\mathscr{C}^{k,\alpha;2}$ then

$$\mathcal{H}_{\overline{g}}(\rho) = \text{traceless } \operatorname{Hess}_{\overline{g}} \rho \quad \text{along } \partial M$$

Build in (SF) to $\rho\Sigma = \rho\phi^{-2}(\mu + \mathcal{D}_g W)$

- Require $\mathcal{H}_{\overline{g}}(\rho)$ to be leading order term of $\rho\mu$
- solve for "correction" $\mathcal{D}_g W$ in more highly weighted space

Summary

Defined $\mathscr{C}^{k,\alpha;m}$ weakly asymptotically hyperbolic metrics

- high interior regularity; limited boundary regularity
- Fredholm theory for associated elliptic operators
- Yamabe and CMC conformal method solution maps close

In the $\mathscr{C}^{k,\alpha;2}$ weakly AH setting we can

- incorporate shear-free condition in to CMC conformal method
- construct (and indeed parametrize) CMC hyperboloidal data

References

- with James Isenberg, John M. Lee, Iva Stavrov Allen:
  - `arXiv:1506.03399`. Weakly asymptotically hyperbolic manifolds, to appear in CAG.
  - `arXiv:1506.06090` The shear-free condition and constant-mean-curvature hyperboloidal initial data, to appear in CQG.

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