Modular Forms for Abelian Varieties

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Modularity of Elliptic Curves

Theorem (Modularity Theorem - Wiles 1994, BCDT 2000)

For an elliptic curve $E/\mathbb{Q}$ of conductor N, there exists a cusp form f of weight 2 and level N such that

$$L(E, s) = L(f, s).$$

(For a complete story of the modularity theorem and Fermat's last theorem, see for example "Modular Forms and Fermats Last Theorem" by Cornell, Silverman, and Stevens.)

Cusp Forms

In this theorem, f is a cusp form for the Hecke's subgroup

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Hence,

$$f : \Gamma_0(N) \backslash \mathbb{H} \to \mathbb{C}.$$

N is called the **level** of such a modular form.

Also, f can be viewed as a function on a double coset space

$$f : \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R}) \to \mathbb{C}$$

where, SL_2 is the special linear group and SO_2 is the special othogonal group.

Using **strong approximation theorem** one may convert f to an automorphic representation ϕ_f of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$:

$$\phi_f : Z_{\mathbb{A}_\mathbb{Q}} \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) / K_\infty \times \prod_p K_p^N \to \mathbb{C},$$

with trivial central character, where

- $Z_{\mathbb{A}_\mathbb{Q}}$ is the center of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$
- $K_\infty = \mathrm{GO}_2(\mathbb{R})$
- $K_p^N = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \mid c \equiv 0 \pmod{N} \right\}.$

Generalization to Abeian Varieties

For abelian varieties, a generalization of such a modularity theorem is attributed to the Langlands program:

For an abelian variety $A/\mathbb{Q}$ of dimension n there exists an automorphic representation ϕ on GSpin_{2n+1} such that

$$L(A, s) = L(\phi, s).$$

Gross' Refinement of This Conjecture

Gross has a refinement of this conjecture for a special case:

Conjecture (Gross, 2015)

Let $A/\mathbb{Q}$ be an abelian variety of dimension n with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. Then there exists an automorphic representation on the split $\mathrm{GSpin}_{2n+1}(\mathbb{A}_{\mathbb{Q}})$,

$$\phi : \mathrm{GSpin}_{2n+1}(\mathbb{Q})\backslash\mathrm{GSpin}_{2n+1}(\mathbb{A}_{\mathbb{Q}})/K_\infty \times \prod_p K_p^N \to \mathbb{C},$$

with explicit wieght and level, such that

$$L(A, s) = L(\phi, s).$$

See "On the Langlands correspondence for symplectic motives", Gross, 2015.

Gross' Refinement to This Conjecture

In that article, Gross works with SO_{2n+1} instead of GSpin_{2n+1}.

In fact, by "renormalizing", one may assume that ϕ lives on SO_{2n+1}:

Conjecture (Gross, 2015)

Let $A/\mathbb{Q}$ be an abelian variety of dimension n with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. Then there exists an automorphic representation on the split $\mathrm{SO}_{2n+1}(\mathbb{A}_{\mathbb{Q}})$,

$$\phi : \mathrm{SO}_{2n+1}(\mathbb{Q})\backslash\mathrm{SO}_{2n+1}(\mathbb{A}_{\mathbb{Q}})/K_\infty \times \prod_p K_0(p^m) \to \mathbb{C},$$

with explicit wiegght and level, such that

$$L(A,s) = L(\phi,s).$$

Split SO_{2n+1} and GSpin_{2n+1}

Gross' conjecture addresses the special orthogonal group $\mathrm{SO}(\Lambda)$ for the $\mathbb{Z}$-lattice

$$\Lambda = \langle a_1, \ldots, a_n, c, b_n, \ldots, b_1 \rangle$$

with bilinear form $(-,-)$ with

$$(a_i, b_i) = 1, \quad (c, c) = 2,$$

and all other inner products equal to zero.

The corresponding **GSpin** group is the group satisfying the short exact sequence

$$1 \to \mathrm{GL}_1 \to \mathrm{GSpin}(\Lambda) \to \mathrm{SO}(\Lambda) \to 1.$$

It is a group of type B_n whose "derived" subgroup $\mathrm{Spin}(\Lambda)$ is the double cover of $\mathrm{SO}(\Lambda)$.

The Level as a Group Scheme

Let $N \in \mathbb{N}$. Let $\Lambda(N)$ be the sublattice spanned by the vectors

$$\{a_1, \ldots, a_n, Nc, Nb_n, \ldots, Nb_1\}$$

over $\mathbb{Z}$. The appropriate bilinear form on this sublattice is $(-,-)/N$.

Gross determines the "level" of the automorphic representation ϕ as a group scheme $K_0(N)$:

Proposition (Gross, 2015)

There exists a group scheme $K_0(N)/\mathbb{Z}$ with generic fiber isomorphic to $\mathrm{SO}_{2n+1}/\mathbb{Q}$ and special fiber at p isomorphic to $\mathrm{SO}_{2n+1}/\mathbb{F}_p$ if $p \nmid N$ and $\mathrm{SO}_{2n}/\mathbb{F}_p$ if $p \mid N$.

Special Cases

- When $n = 1$, $\mathrm{SO}_3 \cong \mathrm{SL}_2$ and $K_0(N)$ is conjugate to

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

- When $n = 2$, $\mathrm{SO}_5 \cong \mathrm{PGSp}_4$ and $K_0(N)$ is conjugate to the "paramodular subgroups" defined by Roberts and Schmidt (see "Local newforms for GSp_4" by Roberts and Schmidt).

My PhD Project So Far

In my thesis, I have used Gross' construction to determine the corresponding level for $\mathrm{GSpin}(\Lambda)$:

Proposition (S.)

There exists a group scheme $G = G(\Lambda(N))/\mathbb{Z}$ *with generic fiber* $G_{\mathbb{Q}} \cong \mathrm{GSpin}_{2n+1}/\mathbb{Q}$ *and special fiber* $G_{\mathbb{F}_p} \cong \mathrm{GSpin}_{2n+1}/\mathbb{F}_p$ *if* $p \nmid N$ *and* $G_{\mathbb{F}_p} \cong \mathrm{GSpin}_{2n}/\mathbb{F}_p$ *if* $p \mid N$ *that satisfies the short exact sequence*

$$1 \to \mathrm{GL}_1/\mathbb{Z} \to G/\mathbb{Z} \to K_0(N)/\mathbb{Z} \to 1.$$

A Cusp form

Gross gives an explicit recipe for constructing a global cusp form F, made from a tensor product of local cusp forms:

$$F : G(\mathbb{Q})\backslash \mathrm{GSpin}_{2n+1}(\mathbb{A}_{\mathbb{Q}})/K_\infty \times \prod_p G(\mathbb{Z}_p) \to \mathbb{C}.$$

By strong approximation, F is completely determined by restricting to its archimedean component

$$F_\infty : G(\mathbb{Q})\backslash \mathrm{GSpin}_{2n+1}(\mathbb{R})/K_\infty \to \mathbb{C}.$$

My Future Work

1. Determine K_∞: Gross has also addressed the weight K_∞ of the automorphic representation ϕ for SO_{2n+1} explicitly. What is the corresponding weight for GSpin_{2n+1}?

2. Gross' work is only focused on the case of trivial central character for the automorphic representation ϕ. What if we had a nontrivial central character? In other words, I am interested in finding a class of GSpin automorphic representations with an arbitrary central character whose restriction to the trivial central character is the work of Gross.

Motivations

Cunningham and Dembélé have recently used lifts of Hilbert modular forms to general odd spin groups to construct nontrivial examples of abelian varieties that satisfy Gross' conjecture:

$$f \xrightarrow{\text{Arthur-Clozel}} \pi' \text{ on } \mathrm{GL}_{2n} \xrightarrow{\text{Shahidi-et } al.} \pi' \text{ descends to } \mathrm{GSpin}_{2n+1}.$$

Thank You!