Representing topological full groups in Steinberg algebras and C*-algebras

(Joint work with L.O. Clark, M. Ghandehari, E.J. Kang, and D. Yang)

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Ample groupoids and inverse semigroups

A topological groupoid is **ample** if it has a basis of compact open bisections ("cobs").

An **inverse semigroup** is a semigroup S such that every element $s \in S$ has a unique **inverse** $s^* \in S$ satisfying $ss^*s = s$ and $s^*ss^* = s^*$.

Setup: Let G be an ample Hausdorff groupoid with compact unit space $G^{(0)} = r(G) = s(G)$.

The cobs in G form an inverse semigroup $S(G)$ with identity $G^{(0)}$ under the operations

$$B_1B_2 = \{\gamma_1\gamma_2 : \gamma_1 \in B_1,\, \gamma_2 \in B_2,\, s(\gamma_1) = r(\gamma_2)\} \quad \text{and} \quad B^* = \{\gamma^{-1} : \gamma \in B\}.$$

Topological full groups

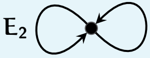
We call a cob B **full** if $r(B) = s(B) = G^{(0)}$. The **topological full group** of G is the group

$$F(G) \coloneqq \{B \in S(G) : B \text{ is full}\} \subseteq S(G) \setminus \{\varnothing\}.$$

We have $F(G) = S(G) \setminus \{\varnothing\}$ if and only if G is a group. In this case, $F(G) = \{\{\gamma\} : \gamma \in G\} \cong G$.

Example

If G_2 is the Cuntz groupoid (i.e. the boundary-path groupoid of E_2), then $F(G_2)$ is Thompson's group V_2, and $C^*(G_2) \cong C^*_r(G_2) \cong \mathcal{O}_2$.



Theorem (Matui 2015)

Let G and H be minimal, effective, ample Hausdorff groupoids such that $G^{(0)}$ and $H^{(0)}$ are Cantor sets. Then $G \cong H \iff F(G) \cong F(H)$.

Steinberg algebras

The **(complex) Steinberg algebra** of G is the $*$-algebra

$$\mathcal{A}(G) \coloneqq \operatorname{span}\{1_B : B \in S(G)\} = \{f \in C_c(G) : f \text{ is locally constant}\},$$

with multiplication satisfying $1_{B_1} * 1_{B_2} = 1_{B_1 B_2}$ and involution satisfying $1_B^* = 1_{B^*}$.

This $*$-algebra is dense in both $C_r^*(G)$ and $C^*(G)$.

Extending the operations on $S(G)$ gives a $*$-algebra of point-mass functions

$$\mathbb{C}S(G) \coloneqq \operatorname{span}\{\delta_B : B \in S(G)\}$$

with $*$-subalgebra $\mathbb{C}F(G) = \operatorname{span}\{\delta_B : B \in F(G)\}$.

What is the relationship between $\mathcal{A}(G)$, $\mathbb{C}S(G)$, and $\mathbb{C}F(G)$?

Representing topological full groups in Steinberg algebras

There is a representation $\widetilde{\pi}\colon \mathbb{C}S(G) \to A(G)$ satisfying $\widetilde{\pi}(\delta_B) = 1_B$ for each $B \in S(G)$.

The representation is **surjective** because $A(G) = \text{span}\{\widetilde{\pi}(\delta_B) : B \in S(G)\}$.

The representation is typically **not injective**: if $B \in S(G)$ satisfies $B = B_1 \sqcup B_2$ for nonempty $B_1, B_2 \in S(G)$, then $1_{B_1} + 1_{B_2} = 1_B$, but $\delta_{B_1} + \delta_{B_2} \neq \delta_B$.

Since $F(G) \subseteq S(G)$, we have $\mathbb{C}F(G) \subseteq \mathbb{C}S(G)$, so $\widetilde{\pi}$ restricts to a representation $\pi\colon \mathbb{C}F(G) \to A(G)$ with image $\text{im}(\pi) = \text{span}\{1_B : B \in F(G)\}$.

Questions: When is π injective? When is $\text{im}(\pi)$ all of $A(G)$?

Injectivity and surjectivity of the representation $\pi\colon \mathbb{C}F(G) \to A(G)$

Theorem (A–Clark–Ghandehari–Kang–Yang 2024)

(a) $\pi\colon \mathbb{C}F(G) \to A(G)$ *is **surjective** if and only if* G *is a group.*

(b) $\pi\colon \mathbb{C}F(G) \to A(G)$ *is **injective** if and only if either*

 (i) $G = \mathrm{Iso}(G) \coloneqq \{\gamma \in G : r(\gamma) = s(\gamma)\}$ *and* G *has at most one nontrivial isotropy group (i.e.* $G =$ *group* $\sqcup$ *other units); or*

 (ii) $G \neq \mathrm{Iso}(G)$ *and* $\left|G \setminus G^{(0)}\right| < 3$ *(i.e.* $G = \{\gamma, \gamma^{-1}\} \sqcup$ *units, where* $r(\gamma) \neq s(\gamma)$*).*

Why is π not injective when (i) and (ii) don't hold?

Idea: There are several cases to consider. For example, suppose $B = B_1 \sqcup B_2 \in F(G)$ for nonempty cobs $B_1, B_2 \subseteq G \setminus G^{(0)}$ with $r(B_i) = s(B_i)$. Then $D_1 \coloneqq B_1 \sqcup r(B_2) \in F(G)$ and $D_2 \coloneqq r(B_1) \sqcup B_2 \in F(G)$, and $1_{D_1} + 1_{D_2} = 1_B + 1_{G^{(0)}}$, but $\delta_{D_1} + \delta_{D_2} \neq \delta_B + \delta_{G^{(0)}}$.

Density of the image of the representation $\pi\colon \mathbb{C}F(G) \to A(G)$

Recall that $A(G)$ is dense in $C^*(G)$ and $C^*_r(G)$. And when G is not a group, $\mathrm{im}(\pi) \neq A(G)$.

Question: When G is not a group, could $\mathrm{im}(\pi)$ still be dense in $C^*(G)$ or $C^*_r(G)$?

Recall that for the Cuntz groupoid G_2, we have $F(G_2) = V_2$, and $C^*(G_2) \cong C^*_r(G_2) \cong \mathcal{O}_2$, and $A(G_2)$ is the Leavitt path algebra L_2.

Theorem (Haagerup–Olesen 2017)

$\pi(\mathbb{C}V_2)$ is dense in $\mathcal{O}_2$ (even though $\pi\colon \mathbb{C}V_2 \to A(G_2)$ is not surjective).

We consider this question further in the setting where G is a **discrete** groupoid with finitely many units.

Density of the image of π in $C^*(G)$ for a discrete groupoid G

Theorem (A–Clark–Ghandehari–Kang–Yang 2024)

Let G be a discrete groupoid with finitely many units. Then $\mathrm{im}(\pi)$ is dense in $C^*(G)$ if and only if G is a group.

Proof strategy.

- Let $n := \left|G^{(0)}\right|$. If $n = 1$, then G is a group and $\mathrm{im}(\pi) = A(G)$ is dense in $C^*(G)$.
- Suppose $n > 1$. For each $\gamma \in G$, let $T\left(1_{\{\gamma\}}\right) \in M_n(\mathbb{C})$ be the $\left(r(\gamma), s(\gamma)\right)$ matrix unit.
- Since each $f \in A(G)$ can be written as $f = \sum_{\gamma \in \mathrm{supp}(f)} f(\gamma) 1_{\{\gamma\}}$, we can extend T linearly to get a representation $T\colon A(G) \to M_n(\mathbb{C})$ that is full-norm-bounded.
- Show that for each $f \in \mathrm{im}(\pi)$, the matrix $T(f)$ has constant row and column sums.
- Since $n > 1$, it follows that $1_{\{\gamma\}} \notin \overline{\mathrm{im}(\pi)}^{\|\cdot\|_{\max}}$ for all $\gamma \in G$, so $\overline{\mathrm{im}(\pi)}^{\|\cdot\|_{\max}} \neq C^*(G)$. ■

Density of the image of π in $C^*_r(G)$ for a discrete groupoid G

The previous theorem does not hold in the reduced setting! To see this, we construct a discrete groupoid G with $2 \leqslant |G^{(0)}| < \infty$ for which $\mathrm{im}(\pi)$ is dense in $C^*_r(G)$.

Example (A–Clark–Ghandehari–Kang–Yang 2024)

- Let $G = \mathbb{F}_2 \sqcup \mathbb{F}_2 = \{(g, i) : g \in \mathbb{F}_2,\ i \in \{1, 2\}\}$. Then $F(G) \cong \mathbb{F}_2 \times \mathbb{F}_2$.
- By symmetry, it suffices to show that $1_{\{(g,1)\}} \in \overline{\mathrm{im}(\pi)}^{\|\cdot\|_r}$ for all $g \in \mathbb{F}_2$.
- Order $\mathbb{F}_2 = \{t_1, t_2, t_3, \dots\}$ by word length. Define $(\phi_n)_{n=1}^{\infty} \subseteq \mathrm{im}(\pi)$ by

$$\phi_n \coloneqq \pi\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{\{(g,1),(t_i,2)\}}\right) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{(g,1),(t_i,2)\}} = 1_{\{(g,1)\}} + \frac{1}{n}\sum_{i=1}^{n} 1_{\{(t_i,2)\}}.$$

- [Haagerup 1978] provides a bound on norms of elements of $C^*_r(\mathbb{F}_2)$, which we use to show that $\frac{1}{n}\sum_{i=1}^{n} 1_{\{(t_i,2)\}} \to 0$ as $n \to \infty$, so $\phi_n \to 1_{\{(g,1)\}}$ in $C^*_r(G)$.

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