Recovering elements of groupoid C^*-algebras from their supports

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joint work with Pradyut Karmakar

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Intermediate algebras

If $D \subseteq A$ is a Cartan subalgebra, then there is a groupoid twist $\Sigma \to G$ where G is an essential, Hausdorff étale groupoid such that

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Theorem (Brown-Exel-F-Pitts-Reznikoff (2021))

Let A be a nuclear C^*-algebra and let $D \subseteq A$ be a C^*-diagonal. Then, if C is a C^*-algebra with $D \subseteq C \subseteq A$, then D is a C^*-diagonal of C.

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If $\Sigma \to G$ is a the twist associated to $D \subseteq A$, then there is a one-to-one correspondence

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$$\longleftrightarrow$$

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Theorem (Brown-Clark-F)

Let G be an amenable, étale groupoid with the property that: if $x, y \in G^{(0)}$ such that $|xGy| > 1$, then $x = y$, $xG = Gy$, and $|xGx| = 2$. Then if $\Sigma \to G$ is a twist, there is a one-to-one correspondence

$$\{H \subseteq G \colon H \text{ a wide open subgroupoid}\}$$

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Further: if there is more to the isotropy than this, then we do not get the correspondence.

How does the correspondence work? And why nuclear?

There exists an injective, norm-decreasing map
$j\colon C_r^*(\Sigma; G) \to C_0(\Sigma; G)$ (Renault).
Given $C_0(G^{(0)}) \subseteq C \subseteq C_r^*(\Sigma; G)$, then

$$H = \{\gamma \in G \colon j(c)(\gamma) \neq 0 \text{ for some } c \in C\}.$$

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A C^*-algebra $A \simeq C_r^*(\Sigma;G)$ is nuclear if and only if G is amenable (Takeishi (2014)).

Theorem (Brown-Exel-F-Pitts-Reznikoff (2021, 2024))

Let $\Sigma \to G$ be a twist with G amenable. Take any $a \in C_r^*(\Sigma;G)$ and let

$$U = \mathrm{supp}(a) = \{\gamma \in G\colon j(a)(\gamma) \neq 0\}.$$

Then $a \in \overline{C_c(\Sigma|_U;U)}^{\|\cdot\|_r}$.

Group actions

If $\Gamma \curvearrowright X$, Γ discrete and $a \in C(X) \rtimes_r \Gamma$. Let $a \sim \sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma$, be the Fourier series. Then

$$\mathrm{supp}(a) = \bigcup_{\gamma \in \gamma} [\mathrm{supp}(a_\gamma) \times \{\gamma\}] \subseteq \Gamma \times X.$$

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In this case, the amenability condition (of Γ or the action $\Gamma \curvearrowright X$), can be replaced with the condition that Γ has the approximation property.

Recovering elements from their Fourier representation...?

Question

What conditions can we put on G, or $\Sigma \to G$, to guarantee that we can recover elements from their supports?
I.e. if $a \in C_r^*(\Sigma; G)$ and $U = \mathrm{supp}(a)$ can I tell if $a \in \overline{C_c(\Sigma|_U; U)}^{\|\cdot\|}$?

Rapid Decay for Groupoids

Definition

Let G be a groupoid. A function $L\colon G \to [0, \infty)$ is a *length function* if

1. $L(x) = 0$ for all $x \in G^{(0)}$;
2. $L(\gamma^{-1}) = L(\gamma)$ for all $\gamma \in G$;
3. $L(\gamma\eta) \leq L(\gamma) + L(\eta)$, when $r(\eta) = s(\gamma)$.

Definition (Hou (2017), Weygandt (2023))

Let G be an étale groupoid with length function L. For each integer $p \geq 0$ define a norm on $C_c(\Sigma; G)$ by

$$\|f\|_{2,p,L} = \sup_{x\in G^{(0)}} \Big(\sum_{s(\gamma)=x} |f(\gamma)|^2(1 + L(\gamma))^{2p}, \sum_{r(\gamma)=x} |f(\gamma)|^2(1 + L(\gamma))^{2p}\Big)^{1/2}$$

Rapid Decay for Groupoids

The twist $\Sigma \to G$ has *rapid decay* if there is a constant $C \geq 0$, $p \geq 0$ such that

$$\|f\|_r \leq C\|f\|_{2,p,L},$$

for all $f \in C_c(\Sigma; G)$. (Weygandt: rapid decay depends only on L and G, not Σ).

Haagerup Property

Definition

A function $\varphi\colon G \to \mathbb{R}$ is a *locally proper negative type function* if

1. $\varphi(x) = 0$ for all $x \in G^{(0)}$;
2. $\varphi(\gamma) = \varphi(\gamma^{-1})$ for all $\gamma \in G$;
3. for each $x \in G^{(0)}$ and $\gamma_1, \dots, \gamma_n$ with $s(\gamma_i) = x$, and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with $\sum \lambda_i = 0$ we have
$$\sum_{i,j} \lambda_i \varphi(\gamma_i \gamma_j^{-1}) \lambda_j \leq 0;$$
4. the function $(\varphi, r, s)\colon G \to \mathbb{R} \times G^{(0)} \times G^{(0)}$ is proper.

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Kwasnewski-Li-Skalski (2022) introduced the Haagerup property for (Fell-bundles over) groupoids. If a groupoid has a locally negative type function then it will satisfy the Haagerup condition. If the groupoid is ample and satisfies the Haagerup property, then it will have a locally proper negative type function.

Haagerup Property

Lemma

If φ is a locally proper negative type function on G then the function $L = \sqrt{\varphi}$ is a length function on G.

Theorem (F-Karmakar)

Let $\Sigma \to G$ be a twist. Suppose G has a locally proper negative type function φ and that G has rapid decay with respect to $L = \sqrt{\varphi}$.
If $a \in C_r^*(\Sigma; G)$ and $U = \mathrm{supp}(a)$, then

$$a \in \overline{C_c(\Sigma|_U; U)}^{\|\cdot\|}.$$

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This result is known for groups and group actions (Bedos-Conti (2013)).

Groups satisfying the conditions

Lots of groups satisfy the above conditions (Brodzki-Niblo survey) e.g.

1. Groups acting on CAT(0) cube complexes have negative definite length function (Niblo-Reeves (2003)) and can have rapid decay with certain conditions (Chatterji-Ruane (2005)). Examples include
    1. $\mathbb{F}_n$;
    2. finitely generated coxeter groups;
    3. some small cancellation groups (Wise).