Non-discrete topological full groups

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Cartan subalgebras in operator algebras and topological full groups

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Motivation exists

For reasons that I don't have time to explain, we want examples of simple groups that are totally disconnected and locally compact (tdlc), not discrete and generated by a compact subset.

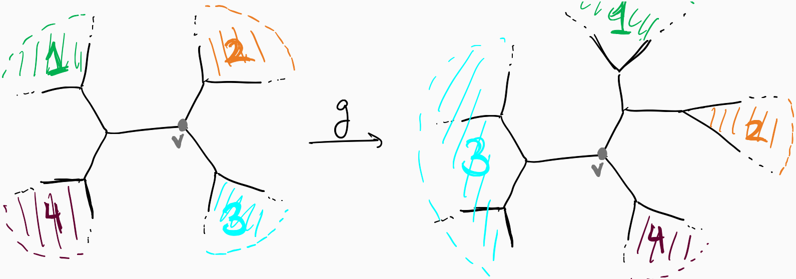
Examples

- $\mathrm{PSL}_n(\mathbb{Q}_p)$, $n \geq 2$, p prime
- $\mathrm{Aut}^+(T)$ =orientation-preserving automorphisms of an infinite tree T ($\neq$ line)

Full groups of Cantor space actions give examples of simple groups .... but these are countable....

E.g., AAut(T) for d-regular tree, $d > 2$, $\partial T = X$ =Cantor space

$\mathrm{AAut}(T) = \{g \in \mathrm{Homeo}(X)$ for which there is a finite clopen partition of $X = \bigsqcup_{i=1}^{n} U_i$ and $g_1, \ldots, g_n \in \mathrm{Aut}(T)$ such that $g \restriction_{U_i} = g_i \restriction_{U_i}, \; i = 1, \ldots, n.\}$



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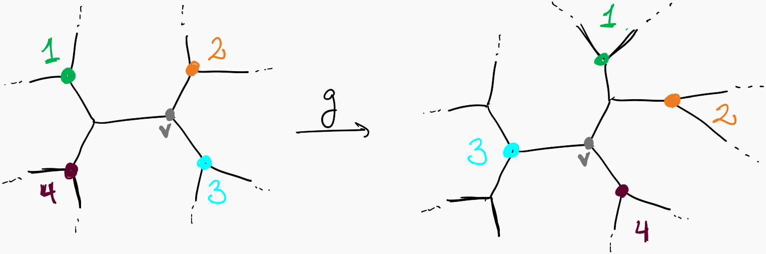

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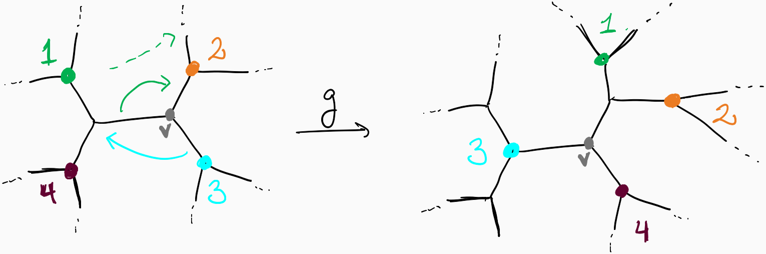

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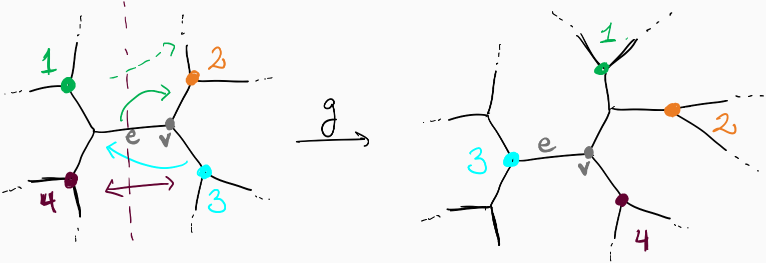

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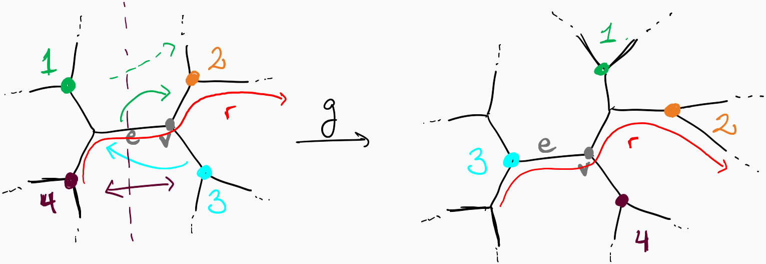

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2: translation along r

Defined in [Neretin, 1984], simplicity in [Kapoudjian, 1994]

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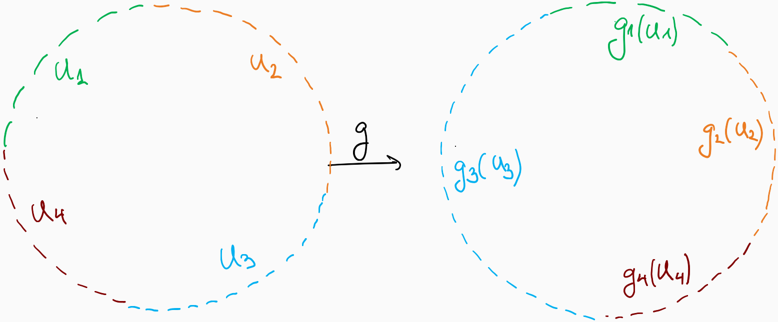

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Topology on $\mathrm{AAut}(T)$

$\mathrm{AAut}(T) \leq \mathrm{Homeo}(X)$, which has a natural group topology: basic opens are

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BUT

$\mathrm{AAut}(T)$ is not closed with respect to this topology!

Instead of top-down, go bottom-up:

$\mathrm{AAut}(T)$ preserves the opens of $\mathrm{St}_{\mathrm{Aut}(T)}(v) \leq_o \mathrm{Aut}(T)$ for any $v \in VT$, i.e.,

$\forall g \in \mathrm{AAut}(T) : \forall U \leq_o \mathrm{St}_{\mathrm{Aut}(T)}(v) : U \cap gUg^{-1} \leq_o \mathrm{St}_{\mathrm{Aut}(T)}(v)$.

Opens of $\mathrm{St}_{\mathrm{Aut}(T)}(v)$ are, for any finite subtree $A \ni v$

$$\mathrm{Fix}(A) = \bigcap_{a \in A} \mathrm{St}(a) = \prod_{\text{leaves of A}} \mathrm{rst}(u)$$

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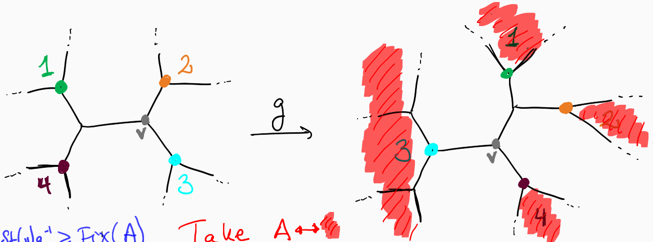

wts $gSt(v)g^{-1} \geq Fix(A)$ Take $A \leftrightarrow$

$k \in \prod rst(\ )$, $k = (k_1, k_2, k_3, k_4)$ $g^{-1}kg = (g_1^{-1}k_1g_1, g_2^{-1}k_2g_2, g_3^{-1}k_3g_3, g_4^{-1}k_4g_4)$

$G = \mathrm{St}_{\mathrm{Aut}(T)}(v)$ is locally decomposable:

for every $U \in \mathcal{CO}(X)$: $\mathrm{rst}_G(U) \times \mathrm{rst}_G(U^\perp)$ is open in G.

This is what guarantees that the topology of $\mathrm{St}_{\mathrm{Aut}(T)}(v)$ can be induced up to a topology on $\mathrm{AAut}(T)$, so that $\mathrm{St}_{\mathrm{Aut}(T)}(v)$ is an open subgroup. As $\mathrm{St}_{\mathrm{Aut}(T)}(v)$ is compact, $\mathrm{AAut}(T)$ is tdlc with this topology.

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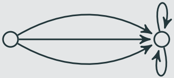
No need to start off with group G to get full group:

$\mathrm{AAut}(T) = F(M)$

$M \leq PHomeo_c(X)$ is the inverse monoid generated by

$\mathrm{St}_{\mathrm{Aut}(T)}(v)$ and 

$\mathcal{BI}(M)$ =Booleanisation of M: take all restrictions of M to $\mathcal{CO}(X)$, allow partial composition, inversion, and finite disjoint joins. $F(M)$ =group of units of $\mathcal{BI}(M)$.

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M is a locally decomposable topological inverse monoid:

- acts continuously on X
- $\forall e \in \mathcal{CO}(X) : M \to Me \times Me^{\perp},\ m \mapsto me \times me^{\perp}$ is continuous and open

Basic open neighbourhoods of $f \in \mathcal{BI}(M)$ are $Nf \cap fN$ for N open neighbourhood of 1 in M.

$\Rightarrow$ topology on $\mathcal{BI}(M)$ has continuous composition, inversion and disjoint joins and makes M open in $\mathcal{BI}(M)$.

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Theorem (G+Reid)

If $M \leq \mathrm{PHomeo}_c(X)$ is locally decomposable inverse monoid, there is a unique inverse monoid topology on $\mathcal{BI}(M)$ that extends that of M and makes taking disjoint joins continuous.

M is tdlc/discrete iff $\mathcal{BI}(M)$ is tdlc/discrete.

Alternating full groups of inverse monoids $M \leq \mathrm{PHomeo}_c(X)$

$A(M) = \langle h \in F(M) : h$ preserves a partition and does an even permutation on the parts $\rangle \leq F(M)$

Theorem (Nekrashevych, '17)

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Theorem (G+Reid)

If M is locally decomposable, compactly generated, expansive, minimal and non-discrete then $\overline{A(M)} = A(M) = F(M)'$ is open in $F(M)$ and simple.

Example: non-discrete Scott-Roever groups

$\Gamma =$closure of Grigorchuk group in Aut(rooted binary tree)

(any other profinite branch group will do)

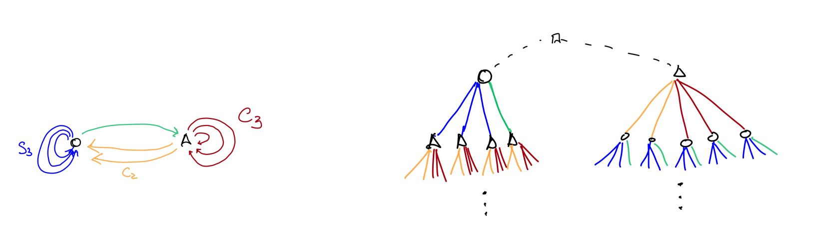
$G = \langle \Gamma, V_2 \rangle$

(Instead of V_2, take a compact subset of $\mathrm{PHomeo}_c(X)$ that preserves the opens of Γ)

Apply theorems

$A(G)$ is simple tdlc group, non-discrete

Example: Coloured Neretin groups [Lederle]



There must be many more!

- non-discrete actions on k-graphs?
- other groupoids that do not obviously come from group actions?
- ...