The ideal intersection property for partial reduced crossed products

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BIRS Cartan Subalgebras in Operator Algebras, and Topological Full Groups

Main Philosophy

Set-up: Discrete group G acting partially on a unital C*-algebra A, $G \curvearrowright A$

Goal: Identify dynamical properties, that give desired C*-algebraic properties, particularly regarding the ideal structure

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Specific question:

If G acts (partially) on A, we obtain a C*-inclusion $A \subseteq A \rtimes_{\mathrm{r}} G$, where $A \rtimes_{\mathrm{r}} G$ encodes A as well as the (partial) dynamics.

We want to find a dynamical property that characterizes when the inclusion has the ideal intersection property (for any $\{0\} \neq J \trianglelefteq A \rtimes_{\mathrm{r}} G$, $A \cap J \neq \{0\}$).

History

Global Action

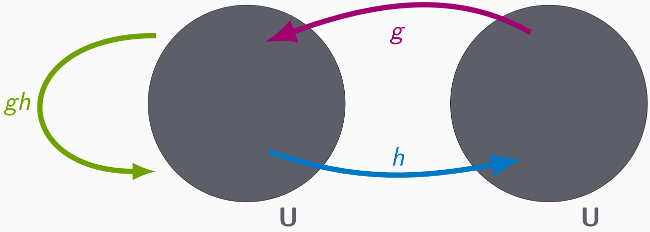
- Kawamura & Tomiyama (1990) give a condition for the IIP to hold in the case of commutative C^*-algebras
- Kennedy and Schafhauser (2019) gave a characterization of IIP for nc crossed products under some conditions

Partial Action

- *Exel, Laca & Quigg (2002)* show that topologically free partial actions on commutative C^*-algebras have the IIP
- *Lebedev (2003)* generalizes this to non-commutative C^*-algebras
- *Giordano & Sierakowski (2014)* give another condition guaranteeing the IIP

Partial Actions on Sets

Global Action: *$G \curvearrowright X$ via a group homomorphism $\alpha\colon G \to \mathrm{Homeo}(X)$.*
$U \subseteq X$ invariant open set, $\alpha_g|_U$ still a group action.



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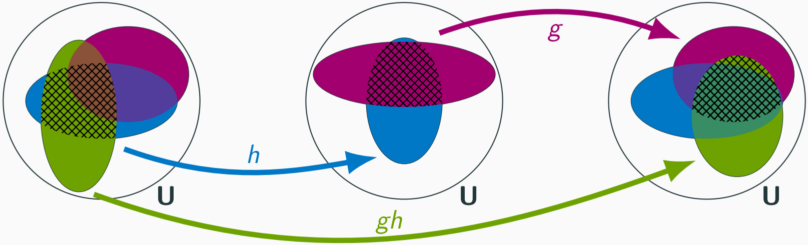
What if U is not invariant?



Partial Actions

Definition

A partial action of a discrete group G on a topological space X is a pair $(\{U_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ where U_g are open sets and $\alpha_g\colon U_{g^{-1}} \to U_g$ are homeomorphisms such that

1. $U_e = X$ and $\alpha_e = \mathrm{id}$
2. $\alpha_g(U_{g^{-1}} \cap U_h) = U_g \cap U_{gh}$
3. $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ whenever $x \in U_{h^{-1}} \cap U_{(gh)^{-1}}$

Fact

Every partial action on a topological space arises from a global action as a restriction.

Partial actions on C*-algebras

For a compact Hausdorff space X using Gelfand duality we obtain the appropriate notion of partial action on a unital C*-algebra A:

$$\begin{aligned} U_g \subseteq X \text{ open } &\rightsquigarrow A_g \trianglelefteq A \\ \alpha_g \colon U_{g^{-1}} \to U_g \text{ homeomorphism } &\rightsquigarrow \alpha_g \colon A_{g^{-1}} \to A_g \text{ *-isomorphism} \\ &\quad \text{(partial *-automorphisms)} \end{aligned}$$

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$$\text{(partial *-automorphisms)}$$

Fact:

Not every partial C*-dynamical system on a unital C*-algebra arises as a restriction of a global action to an ideal!

If each ideal is unital, we can construct a globalization.

Crossed products

Global Action: *From $G \curvearrowright A$ we can build a new C*-algebra $A \rtimes_r G$ generated by A, and G as unitaries implementing the action.*

Similarly, for partial actions we can think of $A \rtimes_r G$ as a particular completion of the covariance algebra

$$\mathcal{C}(\alpha) = \mathrm{span}\{au_s \mid a \in A_s, s \in G\}$$

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Global Action	Partial Action
u_s unitaries	u_s partial isometries
$au_s bu_t = a\alpha_s(b)u_{st}$	$au_s bu_t = \alpha_s(\alpha_{s^{-1}}(a)b)u_{st}$

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Question: How can we give a characterization of the Ideal Intersection Property?

Given a partial action $(\{U_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of G on a compact Hausdorff space X, we can associate a corresponding groupoid

$$\mathcal{G}_\alpha = \{(x, g, y) \in X \times G \times X \mid y \in U_{g^{-1}}, \alpha_g(y) = x\}.$$

Partial Actions as Groupoids

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Note: *Kennedy, Kim, Li, Raum & Ursu (2022)* characterized IIP for essential groupoid C*-algebras for étale groupoids $\mathcal{G}$ with LCH unit space under the assumption that $\mathcal{G}$ is topologically transitive and either Hausdorff or σ-compact.

Expanding our toolbox: Injective Envelopes

Global Actions: Given a group action on a C*-algebra A, we have

$$A \subseteq \boxed{I(A)} \subseteq \boxed{I_G(A)}$$

Rigid – "remembers A", plethora of projections (monotone complete), extensions of u.c.p. maps

Allows for equivariance, injective C*-algebra, monotone complete

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- $A \subseteq A \rtimes_{\mathrm{r}} G$ has the Ideal Intersection Property (IIP)
- $I(A) \subseteq I(A) \rtimes_{\mathrm{r}} G$ has the IIP
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→ Need to construct partial G-injective envelope in appropriate category

What are we working with?

Proposition (Kennedy-K-Sehnem)

A partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$*, induces a unital partial action* $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ *on the injective envelope* $I(A)$*. We have* $I(A)_g = I(A)p_g$ *for a central projection* $p_g \in Z(I(A))$*.*

Additionally, $A \subseteq A \rtimes_{\mathrm{r}} G$ *has the IIP if and only if* $I(A) \subseteq I(A) \rtimes_{\mathrm{r}} G$ *does.*

Up-Shot: We can work with unital partial actions!

Definition

A u.c.p. map $\varphi\colon A \to B$ between is called a *G-morphisms*, if

1. $\varphi(p_g) = q_q$ for all $g \in G$ (where $A_g = Ap_g$, $B_g = Bq_g$),
2. $\varphi(\alpha_g(x)) = \beta_g(\varphi(x))$ for $x \in A_{g^{-1}}$ for all $g \in G$.

If φ is additionally completely isometric, we call φ a *G-embedding*.

Injective Envelopes for Partial Actions

Theorem (Kennedy-K-Sehnem)

A unital partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ *admits a* G*-injective envelope* $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ *in the category of generalized partial actions.*

The G*-injective envelope is unique and* $(I_G(A)_{g\in G}, I_G(\alpha)_{g\in G})$ *is a partial* C^**-dynamical system, i.e.* $I_G(A)_g \trianglelefteq I_G(A)$*.*

Note: If $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is non-unital, define $I_G(A) = I_G(I(A))$.

Non-triviality conditions for partial automorphisms

Goal: We need some non-commutative notion of freeness. Need to ensure that there is no inner part!

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Definition (Kennedy-K-Sehnem)

Consider a partial *-automorphism $\alpha\colon B \to C$ for $B, C \trianglelefteq A$. Then α is inner if there exists a partial isometry $w \in A$ such that $\alpha(x) = wxw^*$ for all $x \in B$. We call α quasi-inner if there exists $w \in I(A)$ such that $\alpha(x) = wxw^*$ for all $x \in B$.

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Good news: The structure of the injective envelope allows us to define this! A partial *-automorphism on $I(A)$ decomposes into an inner and a non-inner part.

Proper outerness

Definition (Proper outerness)

Consider a partial *-automorphism $\alpha\colon B \to C$ for $B, C \trianglelefteq A$. We call α properly outer if the corresponding *-isomorphism $\hat{\alpha}\colon I(B) \to I(C)$ has no inner part.

We call a partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ properly outer if each α_g, $g \neq e$, is properly outer.

Proper outerness by default

Suppose $\alpha\colon B \to C$ for $B, C \trianglelefteq A$ is a *-isomorphism with $I(B) \cap I(C) = \{0\}$. Then α is properly outer.

Conversely: If α is quasi-inner, then $I(B) = I(C)$.

The Result

Theorem (Kennedy-K-Sehnem)

A partial C^*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property if $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is properly outer.

Additionally, if $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has vanishing obstruction, the converse is also true.

Note: Proof relies on equivariant versions of pseudo-expectations.

A counterexample

$G = \mathbb{Z}_3 \times K_4$, where $K_4 = \{e, u, v, uv\}$ denotes the Klein four-group,
$A = M_2 \oplus M_2$, $A_1 := M_2 \oplus 0$ and $A_2 := 0 \oplus M_2$
$\sigma \colon 0 \oplus M_2 \to M_2 \oplus 0 \colon 0 \oplus a \mapsto a \oplus 0$,

$$W_u = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad W_v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad W_{uv} = W_u W_v = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Define the following partial action of G on A via

$$\alpha_{(0,e)} = \mathrm{id}_A, \qquad \alpha_{(0,u)} = \mathrm{Ad}(W_u \oplus W_u), \qquad \alpha_{(0,v)} = \mathrm{Ad}(W_v \oplus W_v)$$
$$\alpha_{(0,uv)} = \mathrm{Ad}(W_{UV} \oplus W_{UV})$$

and

$$\alpha_{(1,h)} \colon A_2 \to A_1 \colon 0 \oplus a \mapsto (\mathrm{Ad}(W_h) \circ \sigma)(0 \oplus a), \text{ for all } h \in K_4,$$
$$\alpha_{(2,h)} \colon A_1 \to A_2 \colon a \oplus 0 \mapsto (\mathrm{Ad}(W_h) \circ \sigma^{-1})(a \oplus 0), \text{ for all } h \in K_4,$$

where we define W_e to be the identity matrix.

Counterexample continued

We have $A \rtimes_{\mathrm{r}} G \cong M_8$, so $A \subseteq A \rtimes_{\mathrm{r}} G$ has the intersection property. However, the system is *not properly outer*.

Issue: The partial isometries implementing the inner parts of the automorphisms can carry an extra "twist", which obstructs the other direction.

Good news: If we exempt cases with this twist we can actually prove the converse, we call this vanishing obstruction.

What does this mean if A is commutative?

Definition

A partial action on a topological space $(\{X_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ is called topologically free if for each $g \neq e$, the fix point set

$$F_g = \{x \in X_{g^{-1}} \mid \theta_g(x) = x\}$$

has empty interior.

Theorem (Kennedy-K-Sehnem)

$\mathcal{C}(X) \subseteq \mathcal{C}(X) \rtimes_{\mathrm{r}} G$ has the IIP if and only if the partial action on the spectrum of $I_G(\mathcal{C}(X))$ is (topologically) free.

Integer Partial actions

Let $I, J \trianglelefteq A$ be regular ideals and $\alpha\colon I \to J$ be a *-isomorphism. We can define a partial action of $\mathbb{Z}$ on A as follows

$$\begin{aligned} A_0 &= A, \\ A_{n+1} &= \{a \in J \mid \alpha^{-1}(a) \in A_n\}, \text{ for } n \geq 0 \\ A_{n-1} &= \{a \in I \mid \alpha(a) \in A_n\}, \text{for } n \leq 0 \end{aligned}$$

Proposition

Suppose $\cap_{n\in\mathbb{Z}} A_n = \{0\}$, then α is properly outer and $A \subseteq A \rtimes_{\mathrm{r}} \mathbb{Z}$ has the ideal intersection property.

Summary

- Existence of a G-injective envelope for partial actions
- A notion of proper outerness for partial actions
- Proper outer partial C^*-dynamical systems have IIP
- Identified an obstruction for the other direction with counterexample
- Characterization in the commutative case
- Examples of integer partial actions