Ample groupoids, topological full groups, algebraic K-theory spectra and infinite loop spaces

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Homology for topological full groups

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- **Theorem** (Szymik-Wahl): $H_*(V_2) \cong \{0\}$ for all $* > 0$.

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- **Idea**:
  - Try to relate group homology of $\boldsymbol{F}(G)$ to groupoid homology of G.
  - This helps because groupoid homology is much more accessible. For example, groupoid homology has been computed for
    - AF groupoids,
    - Transformation groupoids,
    - Tiling groupoids,
    - Graph groupoids,
    - Higher rank graph groupoids,
    - Groupoids of self-similar actions,
    - Groupoids of piecewise affine transformations,
    - ...

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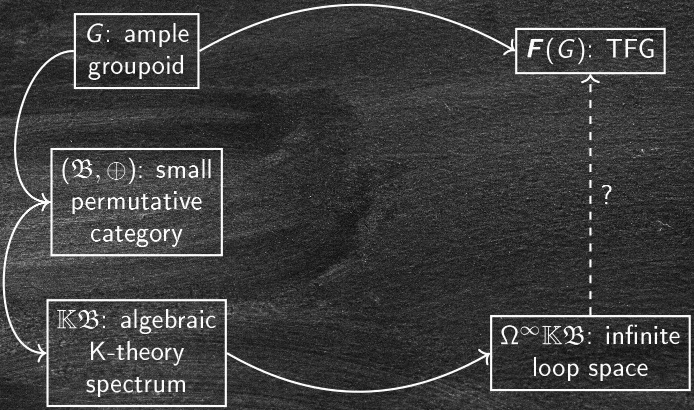

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G has comparison if for all non-empty, compact open $U, V \subseteq G^{(0)}$,

$$\mu(U) < \mu(V) \ \forall\ 0 \neq \mu \in M(G)$$
$$\Rightarrow \exists \text{ compact open bisection } \sigma \subseteq G :\ s(\sigma) = U, r(\sigma) \subseteq V.$$

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Corollary (L): Rational group homology of $\boldsymbol{F}(G)$ is given by $H_*(\boldsymbol{F}(G), \mathbb{Q}) \cong \mathrm{Ext}(H_*^{\mathrm{odd}}(G, \mathbb{Q})) \otimes \mathrm{Sym}(H_*^{\mathrm{even}}(G, \mathbb{Q}))$.

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Corollary (L): The following sequence is exact

$$H_2(\boldsymbol{F}(G)') \to H_2(G) \to H_0(G, \mathbb{Z}/2) \to H_1(\boldsymbol{F}(G)) \to H_1(G) \to 0.$$

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  Now we can define

  $$(U_1, \ldots, U_m) \oplus (V_1, \ldots, V_n) := (U_1, \ldots, U_m, V_1, \ldots, V_n).$$

End

Thank you very much!