Semi-Cartan subalgebras of twisted groupoid C*-algebras

Ying-Fen Lin

Queen's University Belfast
joint work with Tristan Bice, Lisa Orloff Clark, and Kathryn McCormick

BIRS
November 2024

Cartan subalgebras

Cartan pairs in C*-algebras (Renault 2008):

A pair (A, B) of C*-algebras is a *Cartan pair* and say that B is a *Cartan subalgebra* of A if B is a maximal abelian subalgebra of A containing an approximate unit for A, B is regular in A, and there is a faithful conditional expectation of A onto B.

All Cartan pairs arise as twisted groupoid algebras (of effective étale groupoids), and the groupoid and twist are unique.

Definition

A Hausdorff topological groupoid Σ is call a $\mathbb{T}$-groupoid if $\exists$ a free continuous action of $\mathbb{T}$ such that

$$t(ef) = (te)f = e(tf) \quad \text{for all } t \in \mathbb{T}, (e,f) \in \Sigma^{(2)}.$$

A twist is a continuous open groupoid homomorphism $q : \Sigma \twoheadrightarrow G$ from a $\mathbb{T}$-groupoid Σ onto a locally compact Hausdorff étale groupoid G such that $\mathbb{T}$ acts transitively on each fibre, i.e.,

$$q^{-1}(\{q(e)\}) = \mathbb{T}e \text{ for all } e \in \Sigma.$$

- $\exists$ groupoid isomorphism $\iota : \mathbb{T} \times G^{(0)} \twoheadrightarrow q^{-1}(G^{(0)})$ such that

$$\iota(t, q(\boldsymbol{r}(e)))e = te = e\iota(t, q(\boldsymbol{s}(e))) \text{ for all } e \in \Sigma.$$

- Every twist is a proper map (as in K. Courtney, A. Duwenig, M.C. Georgescu, A. an Huef, and M.G. Viola (2024)).
- If $q : \Sigma \twoheadrightarrow G$ is a twist then Σ is locally compact.

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- If $q : \Sigma \twoheadrightarrow G$ is a twist then Σ is locally compact.

Let G be a locally compact Hausdorff étale groupoid and Σ be a twist over G. We say that $f \in C(\Sigma)$ is $\mathbb{T}$-contravariant if

$$f(z \cdot \gamma) = \overline{z} \cdot f(\gamma) \quad \text{for all}\, \gamma \in \Sigma, z \in \mathbb{T}.$$

Define $C(\Sigma; G) := \{f \in C(\Sigma) : f \text{ is } \mathbb{T}\text{-contravariant}\}$ and let $C_c(\Sigma; G) := C(\Sigma; G) \cap C_c(\Sigma)$. Then $C_c(\Sigma; G)$ is a *-algebra under the operations

$$(f * g)(\gamma) := \sum_{\eta \in q(\gamma) G} f(\sigma(\eta)) g(\sigma(\eta)^{-1}\gamma) \quad \text{and} \quad f^*(\gamma) := \overline{f(\gamma^{-1})},$$

where $\sigma : G \to \Sigma$ is a section of q. Then the reduced twisted groupoid C*-algebra

$$C_r^*(\Sigma; G) := \overline{C_c(\Sigma; G)}^{\|\cdot\|_r},$$

where $\|f\|_r := \sup\{\|\pi_x(f)\| : x \in G^{(0)}\}$ and $\pi_x : C_c(\Sigma; G) \to B(L^2(G_x; \Sigma_x))$ is the regular rep at $x \in G^{(0)}$.

Cartan subalgebras (re-formulated):

Let A be a C*-algebra and B be an abelian subalgebra of A. We say that B is a Cartan subalgebra of A if the following conditions are satisfied:

1. B is a maximal abelian subalgebra of A;
2. $\overline{\text{span}}(N(B)) = A$, where $N(B) = \{n \in A : nB^*n \cup n^*Bn \subseteq B\}$; and
3. there exists a faithful conditional expectation $E : A \twoheadrightarrow B$.

Theorem (Renault 2008, Raad 2022)

Let Σ be a twist over an effective locally compact Hausdorff étale groupoid G. Then $(C_r^*(\Sigma; G), C_0(G^{(0)}))$ is a Cartan pair.

Conversely, if (A, B) is a Cartan pair, then there exists a twist $q : \Sigma \twoheadrightarrow G$, where G is an effective locally compact Hausdorff étale groupoid, such that there is an isomorphism from A to $C_r^*(\Sigma; G)$ that maps B onto $C_0(G^{(0)})$.

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Cartan semigroups

Let A be a C*-algebra and N be a subset of A.

Definition (Bice-Clark-L-McCormick)

We call $N \subseteq A$ a Cartan semigroup if

1. N is a $*$-subsemigroup of A with dense span,
2. $B := C^*(N_+)$ is a commutative subsemigroup of N, where $N_+ := \{n^*n : n \in N\}$, and
3. there is a conditional expectation $E : A \twoheadrightarrow B$ such that

$$E(n)n^* \in B \text{ for all } n \in N.$$

In this case, we call B the associated semi-Cartan subalgebra.

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Lemma

- *Every semi-Cartan subalgebra B contains an approximate unit for A (following Brown-Fuller-Pitts-Reznikoff '21).*
- $\mathbb{C}N = N \subseteq N(B)$.

Fix $(p_k)_k$, a sequence of nonzero polynomials, with zero constant terms that converge to 1 uniformly on all compact subsets of $\mathbb{R} \setminus \{0\}$. Then for any $a \in A$, it follows that

$$p_k(aa^*)a = ap_k(a^*a) \to a.$$

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Proposition

Suppose (A, B) satisfies all properties of a Cartan pair except that the expectation E is faithful. Then $N(B)$ forms a Cartan semigroup with associated semi-Cartan subalgebra $B = C^*(N(B)_+)$.

To see that (3) holds: note that, for all $b \in B$ and $n \in N(B)$,

$$E(n)n^*nn^*b = E(n)n^*bnn^* = E(nn^*bn)n^* = E(bnn^*n)n^* = bE(n)n^*nn^*.$$

Likewise, $E(n)n^*(nn^*)^k b = bE(n)n^*(nn^*)^k$ for all $k > 1$, and hence

$$E(n)n^*b = \lim_k E(n)n^*p_k(nn^*)b = \lim_k bE(n)n^*p_k(nn^*) = bE(n)n^*.$$

As B is a MASA, it follows that $E(n)n^* \in B$, showing that $N(B)$ is a Cartan semigroup.

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Example

Take $G = \mathbb{Z}$, recall that $C_r^*(\mathbb{Z}) \cong C(\mathbb{T})$ via Fourier transform. Then $N = \cup_{k\in\mathbb{Z}} \mathbb{C}\delta_k$ is a Cartan semigroup with $B := C^*(N_+) = \mathbb{C}1$ and expectation E given by $E(a) = \varphi(a)1$, where φ is any state on $C(\mathbb{T})$ such that $\varphi(\delta_k) = 0$ for all $0 \neq k \in \mathbb{Z}$.

However, $\mathbb{Z}$ is not effective, thus, B is not a MASA (ABCCLMR (2023)).

In general, let A be a C*-algebra and suppose there is a state $\varphi : A \to \mathbb{C}$ and a unitary $u \in A$ generating A such that $\varphi(u^k) = 0$ for all $0 \neq k \in \mathbb{Z}$.

$\Longrightarrow$ $N = \cup_{k\in\mathbb{Z}} \mathbb{C}u^k$ is a Cartan semigroup with $B = C^*(N_+) = \mathbb{C}1$ and $E(a) = \varphi(a)1$ for all $a \in A$.

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$\implies N = \cup_{k\in\mathbb{Z}}\mathbb{C}u^k$ is a Cartan semigroup with $B = C^*(N_+) = \mathbb{C}1$ and $E(a) = \varphi(a)1$ for all $a \in A$.

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Theorem (BCLM)

Let A be a C^*-algebra containing a Cartan semigroup N with semi-Cartan sublagebra B generated by the positive elements of N, and a stable expectation $E : A \twoheadrightarrow B$. Then $\exists$ an isomorphism

$$\Psi : A \twoheadrightarrow C := \mathrm{cl}(C_c(\Sigma; G)), \text{ a twisted groupoid } C^*\text{-algebra.}$$

If, furthermore, E is faithful then

$$C = \Psi(A) = C_r^*(\Sigma; G).$$

Key steps:

(i) Construct groupoid $G = \{U \subseteq N : U \text{ is an ultrafilter}\}$ with topology generated by $(\mathcal{U}_n)_{n\in N}$, where

$$\mathcal{U}_n := \{U \in G : n \in U\} \text{ and } G^{(0)} = \bigcup_{b\in B} \mathcal{U}_b,$$

it is Hausdorff and étale.

(ii) For any $U \in G$, whenever $m, n \in U$, define

$$m \sim_U n \quad \text{if and only if} \quad \psi_U(E(n^*m)) > 0,$$

where $\psi_U := \langle B \setminus \mathbf{s}(U)\rangle$. Then $\sim_U$ is an equivalence relation and we define $\Sigma := \{[n]_U : n \in U, U \in G\}$ as a topological space with the subbasis

$$\mathcal{B} = \{\mathcal{E}_n^O : O \subseteq \mathbb{T} \text{ is open and } n \in N\}, \text{ where}$$

$$\mathcal{E}_n^O = \{[tn]_U : t \in O, U \in \mathcal{U}_n\},$$

which proven to be a Hausdorff groupoid with $\Sigma^{(0)} = \{[b]_U : b \in B_+, U \in \mathcal{U}_b\}$ and is a twist over G.

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(iii) For any $a \in A$, define

$$\hat{a}([n]_U) := \frac{1}{\sqrt{\psi_U(n^* n)}} \psi_U(E(n^* a)), \quad \forall [n]_U \in \Sigma.$$

We get

- $b \in B \mapsto \hat{b} \in \{f \in C_0(\Sigma; G) : q(\mathrm{supp}^o(f)) \subseteq G^{(0)}\}$ is a C*-isomorphism;
- $n \in N \mapsto \hat{n} \in \{a \in C(\Sigma; G) : q(\mathrm{supp}^o(a))$ is a bisection$\}$ is a semigroup homomorphism;
- $a \in \mathrm{span}(N) \mapsto \hat{a} \in \mathrm{span}(\{a \in C_0(\Sigma; G) : q(\mathrm{supp}^o(a))$ is a bisection$\})$ is a *-isomorphism; and
- an isomorphism $\Psi : A \twoheadrightarrow \mathrm{cl}(C_c(\Sigma; G))$ such that $\hat{a} = j \circ \Psi(a)$.

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THANK YOU!!