Some Generalizations of Cartan Inclusions

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Outline

- A bit of motivation
- Background on Inclusions and their dynamics
- Cartan Inclusions, their groupoid models and other nice properties
- Examples of non-Cartan inclusions

The generalizations I'll discuss are:

- Γ-Cartan inclusions (joint with Brown, Fuller, Reznikoff)
- Weak Cartan Inclusions (joint with Exel)
- Pseudo-Cartan Inclusions

Some Motivation: W^*-Cartan pairs

In 1977, Feldman & Moore defined notion of a Cartan MASA $\mathcal{D} \simeq L^\infty(X, \mu)$ in a W^*-algebra $\mathcal{M} \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ & showed:

a) $\exists$ Borel equiv. relation R on X with countable equiv. classes & a 2-cocycle σ on R s.t.

$$\mathcal{M} \simeq \mathbf{M}(R, \sigma) \;\&\; \mathcal{D} \simeq \mathbf{A}(R, \sigma),$$

where $\mathbf{M}(R, \sigma)$ are "functions on R" & $\mathbf{A}(R, \sigma)$ are the "functions" supported on diag. $\{(x, x) : x \in X\}$;

b) $\exists$ bijection between (unitary) equivalence classes of such $(\mathcal{M}, \mathcal{D})$ and relations (R, σ) (up to Borel iso).

Hugely influential: allows $\mathcal{M}$ to be viewed as a "fancy matrix algebra".

Kumjian ('86) & then Renault ('08) sought a C^*-algebraic version.

Context: Inclusions

Definition

An inclusion is a pair of C^*-algebras $(\mathcal{C}, \mathcal{D})$ with $\mathcal{D} \subseteq \mathcal{C}$ and $\mathcal{D}$ abelian.

In this generality, wild behavior can happen. Here are restrictions which rule out some bad behavior.

Some Non-degeneracy Conditions

1. $(\mathcal{C}, \mathcal{D})$ is unital if $\mathcal{C}$ unital & $\mathcal{D}$ contains the unit of $\mathcal{C}$.
2. $(\mathcal{C}, \mathcal{D})$ has the approximate unit property (AUP) if $\mathcal{D}$ contains an approximate unit for $\mathcal{C}$. (Sometimes called non-degenerate.)
3. $(\mathcal{C}, \mathcal{D})$ is weakly non-degenerate (WND) if
$$\text{Ann}(\mathcal{C}, \mathcal{D}) := \{c \in \mathcal{C} : dc = cd = 0 \ \forall \ d \in \mathcal{D}\} = \{0\}.$$

Easy to see (1)⇒(2)⇒(3).

Also: $(\mathcal{C}, \mathcal{D})$ weakly non-deg. $\Rightarrow (\tilde{\mathcal{C}}, \tilde{\mathcal{D}})$ unital.

Some Notation

For the inclusion $(\mathcal{C}, \mathcal{D})$, *often write* $X = \hat{\mathcal{D}}$ *and for* $d \in \mathcal{D}, x \in X$,

$$\langle d, x \rangle := \hat{d}(x).$$

Normalizers

Definition

An element $v \in \mathcal{C}$ is a normalizer if $v\mathcal{D}v^* \cup v^*\mathcal{D}v \subseteq \mathcal{D}$; write $\mathcal{N}(\mathcal{C}, \mathcal{D})$ for set of all normalizers.

Facts

1. *$\mathcal{N}(\mathcal{C}, \mathcal{D})$ is a $*$-semigroup.*

Also for $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$:

2. *For all $d \in \mathcal{D}$, $vv^*d = dvv^* \in \mathcal{D}$;*
3. *$\overline{vv^*\mathcal{D}}$ and $\overline{v^*v\mathcal{D}}$ are closed ideals of $\mathcal{D}$.*
4. *$vv^*d \mapsto v^*dv$ uniquely extends to a $*$-isomorphism*
$$\theta_v : \overline{vv^*\mathcal{D}} \to \overline{v^*v\mathcal{D}}.$$

Then θ_v "dualizes" to a partial homeo β_v of X with

$$\operatorname{dom}\beta_v = \{x \in X : \langle v^*v, x\rangle \neq 0\}$$
$$\text{range}\,\beta_v = \{x \in X : \langle vv^*, x\rangle \neq 0\}$$

(Sometimes write $\operatorname{dom} v, \text{range}\, v$ instead of $\operatorname{dom}\beta_v, \text{range}\,\beta_v$.)

Have $\mathcal{W} := \{\beta_v : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}$ an inverse semigroup: $\beta_v \circ \beta_w = \beta_{vw}$, $\beta_v^{-1} = \beta_{v^*}$, idempotents are $\{\beta_d : d \in \mathcal{D}\}$. This is the *Weyl semigroup*.

The Weyl Groupoid (inspired by groupoid of germs for $\mathcal{W}$)

- Let $\mathfrak{Y} := \{(v, x) : v \in \mathcal{N}(\mathcal{C}, \mathcal{D}), x \in \operatorname{dom} \beta_v\}$.
- For two elt's of $\mathfrak{Y}$, say $(v_1, x_1) \sim (v_2, x_2)$ iff $x_1 = x_2 := x$, $\exists d_i \in \mathcal{D}$ s.t. $\langle d_i, x\rangle \neq 0$ & $v_1 d_1 = v_2 d_2$. (Equiv. pairs $\implies v_2^* v_1 d_1 = v_2^* v_2 d_2 \in \mathcal{D}$, so $\operatorname{germ}_x(\beta_{v_1}) = \operatorname{germ}_x(\beta_{v_2})$)
- Set $G := \mathfrak{Y}/\sim$. Write $[v, x]$ for equiv class of (v, x).

Fact

Define

- $[v_1, x_1][v_2, x_2] = [v_1 v_2, x_2]$ *if* $\beta_{v_2}(x_2) = x_1$ *&*
- $[v, x]^{-1} = [v^*, \beta_v(x)]$.

With the topology generated by the sets $\Omega(v, U) := \{[v, x] : x \in U^{open} \subseteq \operatorname{dom} \beta_v\}$, *G becomes a locally compact étale topological groupoid, with* $G^{(0)} = X$, *(G not necessarily Hausdorff.)*

G is the Weyl groupoid of $(\mathcal{C}, \mathcal{D})$.

Standing Assumption on Groupoids

Assumption

All groupoids G will be locally compact, étale, with $G^{(0)}$ Hausdorff.

An Equiv Relation on $\mathbb{C} \times \mathfrak{Y}$

For $i = 1, 2$, say $(\lambda_1, v_1, x_1) \sim (\lambda_2, v_2, x_2)$ if $x_1 = x_2 =: x$, $\exists d_i \in \mathcal{D}$ with $\langle d_i, x\rangle \neq 0$, $v_1 d_1 = v_2 d_2$ & $\dfrac{\lambda_1}{\langle d_1, x\rangle} = \dfrac{\lambda_2}{\langle d_2, x\rangle}$.

Put $\mathcal{L} := \{[\lambda, v, x]\}$ set of equiv classes & set

$$[\lambda_1, v_1, \beta_{v_2}(x)][\lambda_2, v_2, x] := [\lambda_1\lambda_2, v_1 v_2, x] \quad \text{and}$$
$$[\lambda, v, x]^* := [\overline{\lambda}, v^*, \beta_v(x)].$$

Topologize with smallest topology making each of the sections, $G \ni [v, x] \mapsto [f(x), v, x], \quad f \in M(\text{dom } v)$ continuous.

Fact

This makes $\mathcal{L}$ into a Fell line bundle over G.

Can also do this using a suitable $*$-semigroup $N \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$.

Fell Line Bundles $\simeq$ Twists

Fell line bundles over G are equivalent to central groupoid extensions

$$\mathbb{T} \times G^{(0)} \stackrel{i}{\hookrightarrow} \Sigma \stackrel{q}{\twoheadrightarrow} G$$

Given a Fell line bundle $\mathcal{L}$ over the groupoid G, call the pair $(\mathcal{L}, G)$ a twist and sometimes write (Σ, G) instead.

We'll use whichever viewpoint is convenient.

When $\mathcal{L}$ and G arise from an inclusion $(\mathcal{C}, \mathcal{D})$ as above, call $(\mathcal{L}, G)$ (or (Σ, G)) the Weyl twist for $(\mathcal{C}, \mathcal{D})$.

The Convolution Algebra

Can construct: a $*$-algebra $C_c(\mathcal{L}, G)$ (with a convolution product) of certain continuous compactly supported sections of $\mathcal{L}$.

$C_c(\mathcal{L}, G)$ admits various C^*-seminorms, and let $C^*_\eta(\mathcal{L}, G)$ be completion of $C_c(\mathcal{L}, G)/N_\eta$ under a C^*-seminorm η satisfying

$$\eta|_{C_c(G^{(0)})} = \|\cdot\|_{C_0(G^{(0)})}.$$

Call such a seminorm nice.

Get an inclusion $(C^*_\eta(\mathcal{L}, G), C_0(G^{(0)}))$.

The Hope

Given the inclusion $(\mathcal{C}, \mathcal{D})$:

- construct Weyl Twist $(\mathcal{L}, G)$ as above;
- form $(C^*_\eta(\mathcal{L}, G), C_0(G^{(0)}))$

HOPE: $(\mathcal{C}, \mathcal{D})$ and $(C^*_\eta(\mathcal{L}, G), C_0(G^{(0)}))$ are closely related, ideally isomorphic.

Regular Inclusions

It is possible for $\mathcal{N}(\mathcal{C}, \mathcal{D}) = \mathcal{D}$, in which case $G = \hat{\mathcal{D}} = G^{(0)}$, and $\mathcal{L} = \mathbb{C} \times G^{(0)}$ is the trivial line bundle.

So if $\mathcal{N}(\mathcal{C}, \mathcal{D})$ is too small, there's not much hope of using $(\mathcal{L}, G)$ to analyze $\mathcal{C}$.

Definition

The inclusion $(\mathcal{C}, \mathcal{D})$ is regular when $\overline{\operatorname{span}}\, \mathcal{N}(\mathcal{C}, \mathcal{D}) = \mathcal{C}$.

Cartan Inclusions

Definition (Renault, following Kumjian & Feldman-Moore)

A regular incl'n $(\mathcal{C}, \mathcal{D})$ is a Cartan pair or a Cartan inclusion if

- $\mathcal{D}$ is a MASA in $\mathcal{C}$; and
- $\exists$ faith. cond. expectation $\mathbb{E} : \mathcal{C} \to \mathcal{D}$.

(Regular MASA incl'ns automatically have AUP.)

A C^*-diagonal is a Cartan incl'n having unique extension prop. (Kumjian's original axioms differ.)

Examples

- If $\mathcal{C} = \mathcal{D}$ abelian, $\Rightarrow (\mathcal{D}, \mathcal{D})$ Cartan.
- $(M_n(\mathbb{C}), \mathrm{Diag}_n(\mathbb{C}))$ (prototype example)
- X – loc. compact T_2, Γ a discrete gp. acting top. freely on X. Then $(C_0(X) \rtimes_r \Gamma,\ C_0(X))$ a Cartan pair.
- Some graph C^*-algebras have Cartan MASAs, e.g. $(\mathcal{O}_n,\ C^*(\{w(S)w(S)^*\}))$ a Cartan pair.

Hope Fufilled for Cartan inclusions

The nicest property of Cartan inclusions is that they're the C^*-version of the W^*-Cartan subalgebras of Feldman-Moore:

Theorem (Renault '08)

- *Let* $(\mathcal{C}, \mathcal{D})$ *be Cartan with Weyl twist* $(\mathcal{L}, G)$. *Then*
$$(\mathcal{C}, \mathcal{D}) \simeq (C_r^*(\mathcal{L}, G), C_0(G^{(0)}))$$
.
- *Suppose G is a Hausdorff, étale, effective groupoid and* $\mathcal{L}$ *a Fell line bundle over G. Then*
$$(C_r^*(\mathcal{L}, G), C_0(G^{(0)}))$$
is a Cartan pair.

Remark: Renault assumed separability of C^*-algebras & 2nd countability for groupoids; those assumptions are unnecessary (Raad '22).

Other nice properties of Cartan Inclusions

Let $(\mathcal{C}, \mathcal{D})$ be Cartan.

- $\mathcal{C}$ unital, nuclear, separable $\Rightarrow$ $\mathcal{C}$ has UCT (Barlak-Li)—this is important for classification theory.
- If $N \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ a $*$-semigp with $\mathcal{D} \subseteq N$ & $\overline{\operatorname{span}} N = \mathcal{C}$, then $\|\cdot\|_{\mathcal{C}}$ is the smallest C^*-norm on $\operatorname{span} N$ (P).
- If $J \trianglelefteq \mathcal{C}$ is regular, then $(\mathcal{C}/J, \mathcal{D}/(\mathcal{D} \cap J))$ is also Cartan. (Brown-Fuller-P-Reznikoff '24)
- If $\mathcal{A}_1, \mathcal{A}_2$ algebras (need not be C^*-algebras) with

$$\mathcal{D} \subseteq \mathcal{A}_i \subseteq \mathcal{C},$$

any isometric isomorphism $\theta : \mathcal{A}_1 \to \mathcal{A}_2$ uniquely extends to a $*$-iso $\tilde{\theta} : C^*(\mathcal{A}_1) \to C^*(\mathcal{A}_2)$ (P, following ideas of Donsig-P, & Zarikian).

We now move away from the Cartan setting. First some examples.

$\exists$ LOTS of reasonable non-Cartan Reg. Inclusions

Examples (of non-Cartan inclusions)

1. Let $\mathcal{C} = C([0,1])$ & $\mathcal{D} = \{f \in \mathcal{C} : f(0) = f(1)\} \simeq C(\mathbb{T})$. Then $(\mathcal{C}, \mathcal{D})$ a regular non-MASA incln', and $\nexists$ conditional expect. of $C([0,1])$ onto $C(\mathbb{T})$.
2. Let $\mathcal{C} = C([-1,1], M_2(\mathbb{C}))$ and

$$\mathcal{D} = \{f \in \mathcal{C} : f(t) \in C^*(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]) \text{ for } t < 0, \quad f(t) \in C^*(\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]) \text{ for } t > 0\}.$$

Then $(\mathcal{C}, \mathcal{D})$ a reg. MASA incl'n, and $\nexists\ \mathbb{E} : \mathcal{C} \to \mathcal{D}$. (Consider $f(t) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ for $-1 \leq t \leq 1$.)

Beyond Cartan Inclusions

Next we discuss some classes of inclusions $(\mathcal{C}, \mathcal{D})$ where

- $\mathcal{D}$ not a MASA and/or
- there is no conditional expectation.

First we consider a setting where the MASA condition is relaxed, but there is a faithful conditional expectation.

Γ-Cartan Inclusions (with Brown, Fuller & Reznikoff)

Suppose Γ is a discrete abelian group such that $\hat{\Gamma}$ acts (strongly) on the C^*-algebra $\mathcal{A}$, with fixed point algebra $\mathcal{A}^{\hat{\Gamma}}$.

$$\hat{\Gamma} \ni \omega \mapsto \alpha_\omega \in \mathrm{Aut}(\mathcal{A}).$$

Call $a \in \mathcal{A}$ homogeneous of degree $t \in \Gamma$ if $\alpha_\omega(a) = \langle \omega, t \rangle\, a$ for $w \in \hat{\Gamma}$.

Definition (Brown-Fuller-P-Reznikoff '21)

The inclusion $(\mathcal{A}, \mathcal{D})$ is called a Γ-Cartan pair if:
(i) $\mathcal{D} \subseteq \mathcal{A}^{\hat{\Gamma}}$, (ii) $(\mathcal{A}, \mathcal{D})$ is regular, & (iii) $(\mathcal{A}^{\hat{\Gamma}}, \mathcal{D})$ is a Cartan pair.

A Non-MASA Example: $\mathbb{T} \curvearrowright C(\mathbb{T})$ (rotation), so $(C(\mathbb{T}), \mathbb{C}I)$ is $\mathbb{Z}$-Cartan.

Fact

Let $\mathcal{N}_h(\mathcal{A}, \mathcal{D})$ denote the homogeneous normalizers. Then $\mathcal{N}_h(\mathcal{A}, \mathcal{D})$ is a $*$-semigroup & $\overline{\mathrm{span}}\, \mathcal{N}_h(\mathcal{A}, \mathcal{D}) = \mathcal{A}$.

Theorem (BFPR)

1. *Let* $(\mathcal{A}, \mathcal{D})$ *be* Γ*-Cartan. The Weyl twist constructed using* $\mathcal{N}_h(\mathcal{A}, \mathcal{D})$ *is Hausdorff and* $\exists$ *continuous homomorphisms* $c_\Sigma : \Sigma \to \Gamma$ *and* $c_G : G \to \Gamma$ *and a "graded twist"*

$$\begin{array}{ccccc} \mathbb{T} \times G^{(0)} & \longrightarrow & \Sigma & \longrightarrow & G \\ & & & \searrow^{c_\Sigma} & \downarrow c_G \\ & & & & \Gamma. \end{array} \tag{1}$$

2. *Every graded twist as in* (1) *yields a* Γ*-Cartan inclusion* $(C^*_r(\Sigma, G), C_0(G^{(0)}))$.
3. *If the graded twist* (Γ, G) *arises from a* Γ*-Cartan inclusion* $(\mathcal{A}, \mathcal{D})$*, then* $(\mathcal{A}, \mathcal{D}) \simeq (C_r(\Sigma, G), C_0(G^{(0)}))$.

When there is no conditional expectation, the situation can get complicated.

An Issue for non-Cartan Regular MASA Inclusions

Theorem (P, maybe others?)

Let $(\mathcal{C}, \mathcal{D})$ *be a (unital) regular MASA inclusion. Then the Weyl groupoid G is Hausdorff if and only if there exists a conditional expectation* $E : \mathcal{C} \to \mathcal{D}$.

Moral: For regular MASA inclusions, failure of Cartan due to lack of conditional expectation causes inconvenience: groupoid models are not T_2.

What to do when Weyl Groupoid not Hausdorff?

Accept it: seek groupoid models for $(\mathcal{C}, \mathcal{D})$ using non-Hausdorff groupoids

Fight it: seek a Cartan inclusion $(\mathcal{A}, \mathcal{B})$ closely related to $(\mathcal{C}, \mathcal{D})$.

Two of the generalizations of Cartan inclusions we'll discuss take these approaches.

Acceptance. Weak Cartan Inclusions (with Exel)

Weak Cartan inclusions: a setting where there are groupoid models with possibly non-Hausdorff groupoids

Background: Free Points & Smooth Elements

Let $(\mathcal{C}, \mathcal{D})$ be an inclusion, & put $X = \hat{\mathcal{D}}$ (the pure states on $\mathcal{D}$).

Definitions

- $x \in X$ is free if x uniquely extends to a state ϕ_x on $\mathcal{C}$. Let $F = \{\text{free points}\}$.
- If F is dense in X, $(\mathcal{C}, \mathcal{D})$ is a topologically free inclusion.

Kumjian's C^*-diagonals have $F = X$ (by hypothesis), & (separable) Cartan inclusions are topologically free.

Definitions

Let $c \in \mathcal{C}$.

- $x \in X$ is free relative to c if ϕ_i $(i = 1, 2)$ are states on $\mathcal{C}$ extending x, then $\phi_1(c) = \phi_2(c)$. Put $F_c = \{\text{free points relative to } c\}$.
- c is smooth if $\text{int}(F_c)$ is dense in X.

FACT: $(\mathcal{C}, \mathcal{D})$ a MASA incl'n $\Rightarrow$ every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ is smooth.

More Definitions: Smooth Inclusions & the Grey Ideal

Definition

$(\mathcal{C}, \mathcal{D})$ is a smooth inclusion if $\exists$ a $*$-semigroup $N \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ such that: each $v \in N$ is smooth and N is generating i.e. the $\mathcal{D}$-bimodule generated by N is dense in $\mathcal{C}$.

Reg. MASA inclusions are smooth: can take $N = \mathcal{N}(\mathcal{C}, \mathcal{D})$.

Theorem

Let $(\mathcal{C}, \mathcal{D})$ be a reg. incl'n and put

$$\mathfrak{G} := \{c \in \mathcal{C} : \phi_x(c^*c) = 0 \,\forall\, x \in F\}.$$

Then $\mathfrak{G}$ is a closed 2-sided ideal in $\mathcal{C}$, the grey ideal.
In addition, if $(\mathcal{C}, \mathcal{D})$ is topologically free, then

1. $\mathfrak{G} \cap \mathcal{D} = \{0\}$;
2. *if $J \subseteq \mathcal{C}$ an ideal with $J \cap \mathcal{D} = \{0\}$, then $J \subseteq \mathfrak{G}$.*

Weak Cartan Inclusions

Definition

The inclusion $(\mathcal{C}, \mathcal{D})$ is a weak Cartan inclusion if it has the AUP and is a smooth inclusion with vanishing grey ideal.

- The vanishing of $\mathfrak{G}$ is a replacement for faithfulness of the conditional expectation of Cartan inclusions.
- The examples above are weak Cartan inclusions.

Facts: For an inclusion $(\mathcal{C}, \mathcal{D})$ with AUP,

- *weak Cartan* $\Rightarrow$ *regular (by def'n) & topologically free.*
- *smooth + top. free* $\Rightarrow$ $(\mathcal{C}/\mathfrak{G}, \mathcal{D})$ *is weak Cartan.*

When $\mathcal{C}$ is separable, $(\mathcal{C}, \mathcal{D})$ Cartan $\implies$ weak-Cartan (not always true in non-separable case).

A Groupoid Model for Weak Cartan Inclusions

Theorem (Exel-P)

Let $(\mathcal{C}, \mathcal{D})$ be an inclusion, with $\mathcal{C}$ separable.

1. *Suppose $(\mathcal{C}, \mathcal{D})$ is a weak Cartan inclusion & N is a generating $*$-semigroup of smooth normalizers. Then*
   (a) *the Weyl twist $(\mathcal{L}_N, G_N)$ satisfies: G_N is 2nd countable & topologically free.* [a]
   (b) *There is a nice semi-norm $\|\cdot\|_{ess}$ on $C_c(\mathcal{L}_N, G_N)$ such that*
   $$(\mathcal{C}, \mathcal{D}) \simeq (C^*_{ess}(\mathcal{L}_N, G_N), C_0(G^{(0)})).$$
2. *If $(\mathcal{L}, G)$ is a twist as in 1(a), $(C^*_{ess}(\mathcal{L}, G), C_0(G^{(0)}))$ is a weak-Cartan inclusion.*

[a] G is topologically free if $\{x \in G^{(0)} : \text{isotropy at } x \text{ is trivial}\}$ is dense in $G^{(0)}$.

Fight. Pseudo-Cartan Inclusions

We start by reframing Cartan inclusions.

Equivalent Description of Cartan Inclusions

Let $\mathcal{B} \subseteq \mathcal{A}$ be C^*-algebras. Use

$$\mathcal{B}^c := \mathcal{A} \cap \mathcal{B}' \text{ (relative commutant).}$$

Also, say $\mathcal{B} \subseteq \mathcal{A}$ has the ideal intersection property (iip) if $J \trianglelefteq \mathcal{A} \ \& \ J \cap \mathcal{B} = \{0\} \implies J = \{0\}$.

Standard Axioms	**Alternate Axioms**
regular	regular
MASA	$\mathcal{D}^c$ abelian & iip for $\underbrace{\mathcal{D} \subseteq \overbrace{\mathcal{D}^c \subseteq \mathcal{C}}^{iip}}_{iip}$
$\exists$ faithful cond. expect $E : \mathcal{C} \to \mathcal{D}$	$\exists$ cond. expect $E : \mathcal{C} \to \mathcal{D}$

Fact [P '21]: The incl'n $(\mathcal{C}, \mathcal{D})$ is Cartan iff either set of axioms hold.

Pseudo-Cartan Inclusions

Large families of non-Cartan regular inclusions arise if we omit the last of the alternate axioms.

Definition (P)

An inclusion $(\mathcal{C}, \mathcal{D})$ is a pseudo-Cartan inclusion if

1. it is regular; and
2. $\mathcal{D}^c$ is abelian & both $\mathcal{D} \subseteq \mathcal{D}^c$ and $\mathcal{D}^c \subseteq \mathcal{C}$ have the iip.

- While Cartan inclusions have the AUP, a pseudo-Cartan inclusion may not.
- However, every pseudo-Cartan inclusion is weakly non-degenerate.

Examples of Pseudo-Cartan Inclusions

Examples

1. Every Cartan inclusion is a pseudo-Cartan inclusion.
2. Every virtual Cartan inclusion, i.e. a reg. MASA inclusion with iip for $\mathcal{D} \subseteq \mathcal{C}$, is a pseudo-Cartan. (Introduced by P ('17), further studied by Taylor ('23) under name "essential Cartan inclusion".)
3. Every weak-Cartan inclusion is a pseudo-Cartan inclusion.
4. The regular non-Cartan examples given above are all pseudo-Cartan inclusions.
5. If X, Y loc. compact Hausdorff & $C_0(X) \subseteq C_0(Y)$ has the ideal intersection property, then $(C_0(Y), C_0(X))$ is pseudo-Cartan.

   In particular, $(C_b(X), C_0(X))$ is a pseudo-Cartan without AUP.

Expansions

Approach: Try to embed a reg. incl'n into a Cartan inclusion.

Definition

Let $(\mathcal{C}, \mathcal{D})$ be an inclusion.

1. An expansion of $(\mathcal{C}, \mathcal{D})$ is a triple $(\mathcal{A}, \mathcal{B} : \alpha)$ with $(\mathcal{A}, \mathcal{B})$ an incl'n & $\alpha : \mathcal{C} \to \mathcal{A}$ a $*$-monomorphism satisfying $\alpha(\mathcal{D}) \subseteq \mathcal{B}$.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\alpha} & \mathcal{A} \\ \subseteq \uparrow & & \uparrow \subseteq \\ \mathcal{D} & \xrightarrow[\alpha|_{\mathcal{D}}]{} & \mathcal{B} \end{array}$$

2. Call $(\mathcal{A}, \mathcal{B} : \alpha)$ a regular expansion if $\alpha : \mathcal{C} \to \mathcal{A}$ is a regular map, i.e. $\alpha(\mathcal{N}(\mathcal{C}, \mathcal{D})) \subseteq \mathcal{N}(\mathcal{A}, \mathcal{B})$.

Embedding Theorem

Embedding Theorem (P. 2012)

For a unital regular inclusion $(\mathcal{C}, \mathcal{D})$*, TFAE*

1. $(\mathcal{C}, \mathcal{D})$ *has a regular expansion* $(\mathcal{A}, \mathcal{B} : \alpha)$ *with* $(\mathcal{A}, \mathcal{B})$ *Cartan;*
2. $(\mathcal{C}, \mathcal{D})$ *has a regular expansion* $(\mathcal{A}, \mathcal{B} : \alpha)$ *with* $(\mathcal{A}, \mathcal{B})$ *a* C^**-diagonal;*
3. *a certain ideal,* $\text{Rad}(\mathcal{C}, \mathcal{D}) \trianglelefteq \mathcal{C}$*, vanishes.*

A compatible state for $(\mathcal{C}, \mathcal{D})$ is a state with $|\rho(v)|^2 \in \{0, \rho(v^* v)\} \; \forall \, v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}$ &

$$\text{Rad}(\mathcal{C}, \mathcal{D}) := \{x \in \mathcal{C} : \rho(x^* x) = 0 \; \forall \, \text{compatible states}\}.$$

Issues

While the embedding theorem is pleasant, there are problems:

- there may be many unrelated Cartan inclusions into which $(\mathcal{C}, \mathcal{D})$ regularly embeds;
- the Cartan inclusion constructed by the theorem may have little to do with the original incl'n.

Desirable types of expansions

The expansion $(\mathcal{A}, \mathcal{B} : \alpha)$ *is*

- *a Cartan expansion if* $(\mathcal{A}, \mathcal{B})$ *Cartan.*
- *a package if it is regular and* $\exists$ *faithful cond. expectation* $\Delta : \mathcal{A} \to \mathcal{B}$ *& image of* $(\mathcal{C}, \mathcal{D})$ *generates* $(\mathcal{A}, \mathcal{B})$, *i.e.*

  $$\mathcal{B} = C^*(\Delta(\alpha(\mathcal{C}))) \quad \text{and} \quad \mathcal{A} = C^*(\alpha(\mathcal{C}), \mathcal{B})$$

  (we're not assuming $(\mathcal{A}, \mathcal{B})$ *is Cartan here);*
- *an envelope if it is a package &* $\alpha(\mathcal{D}) \subseteq \mathcal{B}$ *has the ideal intersection property;*
- *a Cartan envelope if an envelope & Cartan.*

Some Properties of Cartan Envelopes

Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion and suppose $(\mathcal{A}, \mathcal{B} : \alpha)$ is a Cartan envelope for $(\mathcal{C}, \mathcal{D})$.

Facts

- $\mathcal{C}$ *is simple if and only if* $\mathcal{A}$ *is simple.*
- $\mathcal{C}$ *is separable if and only if* $\mathcal{A}$ *is separable.*

It would be nice to know about nuclearity:

Conjecture

$\mathcal{C}$ *is nuclear if and only if* $\mathcal{A}$ *is nuclear.*

Next goal: Characterize the regular inclusions having a Cartan envelope.

Injective Envelopes for Abelian C^*-algebras

Recall: For any C^*-algebra $\mathcal{A}$:

- the local multiplier algebra is

$$M_{loc}(\mathcal{A}) = \varinjlim_{\substack{J \trianglelefteq \mathcal{A}, \\ J \text{ essential}}} M(J),$$

  where ideals are partially ordered by reverse inclusion.

  Since $\mathcal{A} \trianglelefteq \mathcal{A}$, get $\mathcal{A} \subseteq M_{loc}(\mathcal{A})$.
- When $\mathcal{A}$ is abelian, $M_{loc}(\mathcal{A})$ is the injective envelope for $\mathcal{A}$ (Frank-Paulsen). Also, $\mathcal{A} \subseteq M_{loc}(\mathcal{A})$ has iip.

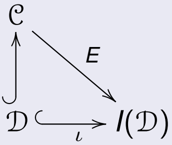
For $\mathcal{A}$ abelian, write $I(\mathcal{A})$ instead of $M_{loc}(\mathcal{A})$, and use $\iota : \mathcal{A} \to I(\mathcal{A})$ for the inclusion map.

Pseudo-Expectations

We saw examples of reg. inclusions with no cond. expectation.
Replacement: Pseudo-expectations.

Definition

A pseudo-expectation for the inclusion $(\mathcal{C}, \mathcal{D})$ is a contractive completely positive linear map $E : \mathcal{C} \to I(\mathcal{D})$ with $E|_{\mathcal{D}} = \iota$.



Let $\mathrm{PsE}(\mathcal{C}, \mathcal{D})$ be the set of all pseudo-expectations for $(\mathcal{C}, \mathcal{D})$.

Since $I(\mathcal{D})$ injective, pseudo-expectations always exist.

Unique Pseudo-Expectations

Usually $\text{PsE}(\mathcal{C}, \mathcal{D})$ is large: when $\mathcal{C}$ unital, $\text{PsE}(\mathcal{C}, \mathbb{C}I) = \mathcal{S}(\mathcal{C})$.

Definition

Let $(\mathcal{C}, \mathcal{D})$ be an inclusion.

1. $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property (!PsE) if $\text{PsE}(\mathcal{C}, \mathcal{D})$ is a singleton set.
2. If $(\mathcal{C}, \mathcal{D})$ has !PsE and the Ps.E. is faithful, it has the faithful unique pseudo-expectation property (f!PsE).

Theorem (P)

Every regular MASA inclusion has !PsE.

Examples show the pseudo-expectation for a regular MASA inclusion need not be faithful.

Theorem

For a regular inclusion $(\mathcal{C}, \mathcal{D})$, *TFAE*

1. $(\mathcal{C}, \mathcal{D})$ *has a Cartan envelope;*
2. $(\mathcal{C}, \mathcal{D})$ *is a pseudo-Cartan inclusion.*
3. $(\mathcal{C}, \mathcal{D})$ *has f!PsE;*

When any of (1)–(3) hold, have:

Uniqueness *If for* $i = 1, 2$, $(\mathcal{A}_i, \mathcal{B}_i : \alpha_i)$ *are Cartan envelopes for* $(\mathcal{C}, \mathcal{D})$, $\exists!$ $*$*-isomorph* $\psi : (\mathcal{A}_1, \mathcal{B}_1) \to (\mathcal{A}_2, \mathcal{B}_2)$ *s.t.* $\psi \circ \alpha_1 = \alpha_2$.

Minimality *If* $(\mathcal{A}, \mathcal{B} : \alpha)$ *is a Cartan pkg for* $(\mathcal{C}, \mathcal{D})$, $\exists$ *ideal* $J \trianglelefteq \mathcal{A}$ *s.t.* $J \cap \alpha(\mathcal{C}) = \{0\}$ *and* $(\mathcal{A}/J, \mathcal{B}/(J \cap \mathcal{B}) : q \circ \alpha)$ *is a Cartan env. for* $(\mathcal{C}, \mathcal{D})$; *(q is quot. map).*

1. For $(C([0,1]), C(\mathbb{T}))$, Cartan env. is $(C([0,1]), C([0,1]) : \text{inclusion map})$.
2. The Cartan envelope for $\mathcal{C} = C([-1,1]), M_2(\mathbb{C}))$ and

$$\mathcal{D} = \{f \in \mathcal{C} : f(t) \in C^*(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])) \text{ for } t < 0,$$
$$f(t) \in C^*(\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right])) \text{ for } t > 0\}$$

is $(\mathcal{A}, \mathcal{B} : \alpha)$ where:

$$\mathcal{A} = C([-1,0], M_2) \oplus C([0,1], M_2)$$
$$\mathcal{B} = C([-1,0], C^*(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])) \oplus C([0,1], C^*(\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]))$$
$$\alpha(f) = f|_{[-1,0]} \oplus f|_{[0,1]} \qquad (f \in \mathcal{C}).$$

Building Pseudo-Cartan Incl'ns from a Cartan Incl'n

Let $(\mathcal{A}, \mathcal{B})$ be Cartan with cond. exp. $\Delta : \mathcal{A} \to \mathcal{B}$. Suppose

- $\mathcal{D} \subseteq \mathcal{B}$ has the iip; and
- $\mathcal{C} := \overline{\text{span}}\, \mathcal{N}(\mathcal{A}, \mathcal{D})$. ($\mathcal{C}$ can be smaller than $\mathcal{A}$.)

Theorem

$(\mathcal{C}, \mathcal{D})$ *is a pseudo-Cartan inclusion. Also, if*

$$\mathcal{B}' = C^*(\Delta(\mathcal{C})) \quad \textit{and} \quad \mathcal{A}' := C^*(\mathcal{B}' \cup \mathcal{C}),$$

then $(\mathcal{A}', \mathcal{B}' :\subseteq)$ *is a Cartan envelope for* $(\mathcal{C}, \mathcal{D})$.

Remarks:

- $\mathcal{N}(\mathcal{A}, \mathcal{D})$ may have little relationship to $\mathcal{N}(\mathcal{A}, \mathcal{B})$;
- $\mathcal{B}' \subseteq \mathcal{B}$, and may be a proper inclusion.

Two (maybe naive) Questions

By the embedding theorem, every unital pseudo-Cartan inclusion regularly embeds into a C^*-diagonal. Not all Cartan inclusions are C^*-diagonals, so we ask:

Question 1

Given a pseudo-Cartan inclusion $(\mathcal{C}, \mathcal{D})$ *with Cartan envelope* $(\mathcal{A}, \mathcal{B} : \alpha)$*, when must* $(\mathcal{A}, \mathcal{B})$ *be a* C^**-diagonal (and not just Cartan)?*

There's been recent interest in finding Cartan MASAs in C^*-algebras.

Question 2

Suppose $(\mathcal{C}, \mathcal{D})$ *is pseudo-Cartan. Must* $\mathcal{C}$ *contain a Cartan MASA? If so, is there a construction?*

A Groupoid Description for the Cartan Envelope

The Twist for the Cartan Envelope

Suppose $(\mathcal{C}, \mathcal{D})$ is a pseudo-Cartan inclusion and let $(\mathcal{A}, \mathcal{B} : \alpha)$ be a Cartan envelope for $(\mathcal{C}, \mathcal{D})$.

Goal

Describe the central extension

$$\mathbb{T} \times G^{(0)} \hookrightarrow \Sigma \twoheadrightarrow G$$

associated to $(\mathcal{A}, \mathcal{B})$ *in terms of data coming as directly from* $(\mathcal{C}, \mathcal{D})$ *as possible.*

Idea: Follow Kumjian's approach to the twist for C^*-diagonals, using the pseudo-expectation instead of the (possibly non-existant) conditional expectation.

Let $E : \mathcal{C} \to I(\mathcal{D})$ be the pseudo-expectation.

1. The functionals
$$\{\sigma \circ E : \sigma \in \widehat{I(\mathcal{D})}\}$$
form the unit space, $G^{(0)} \simeq \Sigma^{(0)}$.
2. For $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, $f \in G^{(0)}$, such that $f(v^*v) \neq 0$,& $c \in \mathcal{C}$, put
$$[v, f](c) := \frac{f(v^*c)}{f(v^*v)^{1/2}}$$
and let
$$\Sigma := \{[v, f] : v \in \mathcal{N}(\mathcal{C}, \mathcal{D}), f \in G^{(0)}, f(v^*v) \neq 0\} \subseteq \mathcal{C}^{\#}.$$
3. $G = \{|\phi| : \phi \in \Sigma\}$

Equip with topology of pointwise convergence. Get twist
$$\mathbb{T} \times G^{(0)} \overset{i}{\hookrightarrow} \Sigma \overset{q}{\twoheadrightarrow} G.$$

Note: G is Hausdorff.

Theorem

Suppose $(\mathcal{C}, \mathcal{D})$ *is a pseudo-Cartan inclusion. As above, form central extension*

$$\mathbb{T} \times G^{(0)} \hookrightarrow \Sigma \twoheadrightarrow G.$$

Define a "Gelfand" map, $\mathfrak{g} : \mathcal{C} \to C(\Sigma)$ *by*

$$\mathfrak{g}(c)(\phi) := \phi(c) \quad c \in \mathcal{C}, \phi \in \Sigma.$$

Let $\mathcal{C}_c := \{c \in \mathcal{C} : \operatorname{supp}(\mathfrak{g}(c))$ *is compact*$\}$ *(a dense* $*$*-subalg). Then* $\mathfrak{g}|_{\mathcal{C}_c} : \mathcal{C}_c \to C_c(\Sigma)$ *is a* $*$*-monomorphism which extends to uniquely to a* $*$*-monomorphism*

$$\alpha : \mathcal{C} \to C_r^*(\Sigma, G)$$

and $(C_r^*(\Sigma, G), C_0(G^{(0)}) : \alpha)$ *is a Cartan envelope for* $(\mathcal{C}, \mathcal{D})$*.*

A Question

Suppose: G is a 2nd-countable, topologically free, not-necessarily Hausdorff groupoid.

- Given a Fell line bundle $\mathcal{L}$ over G, form the weak-Cartan inclusion $(C^*_{\text{ess}}(\mathcal{L}, G), C_0(G^{(0)}))$. This is a pseudo-Cartan inclusion.
- Let $(\mathcal{A}, \mathcal{B} : \alpha)$ be a Cartan envelope for $(C^*_{\text{ess}}(\mathcal{L}, G), C_0(G^{(0)}))$. Let $(\mathcal{L}_{\mathcal{A}}, G_{\mathcal{A}})$ be the Weyl twist for the Cartan inclusion $(\mathcal{A}, \mathcal{B})$.

Question 3

What, if any, is the relationship between $(\mathcal{L}, G)$ and $(\mathcal{L}_{\mathcal{A}}, G_{\mathcal{A}})$? Does $G_{\mathcal{A}}$ depend on the choice of $\mathcal{L}$?

THANK YOU!