Cartan subalgebras from the von Neumann algebra point-of-view

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Cartan subalgebras and equivalence relations

Problems about vN algebraic Cartan subalgebras

Problems solved with vN algebraic Cartan subalgebras

Open problems about vN algebraic Cartan subalgebras

Von Neumann algebras

Definition

A von Neumann algebra is a *-subalgebra $M \subseteq \mathcal{B}(H)$ satisfying $M'' = M$, where H is a complex Hilbert space.

Here $S' = \{a \in \mathcal{B}(H) \mid \forall b \in S\colon ab = ba\}$ is the commutant.

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Definition

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Cartan subalgebras in von Neumann algebras

Definition

A Cartan subalgebra of a von Neumann algebra M is a maximal abelian self-adjoint subalgebra $A \subseteq M$ with a normal, faithful, conditional expectation $\mathrm{E}\colon M \to A$ such that $M = \mathcal{N}_M(A)''$ where $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ is the unitary normaliser.

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Alternatives to the unitary normaliser:

- the partial normaliser consisting of partial isometries normalising A, and
- the stable normaliser of all elements normalising A.

Examples from the group-measure space construction

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Given a discrete group Γ acting non-singularly on a measurable space $(X, [\mu])$, the group-measure space construction $\mathrm{L}^\infty(X) \rtimes \Gamma$ is the (up to isomorphism) unique von Neumann algebra M which

- is generated by a copy of $\mathrm{L}^\infty(X) \subseteq M$ together with
- a multiplicative family of unitaries $(u_g)_{g \in \Gamma}$ satisfying $u_g f u_g^* = {}^g f$ for all $g \in \Gamma$, $f \in \mathrm{L}^\infty(X)$, and
- there is a normal conditional expectation $\mathrm{E} \colon M \longrightarrow A$ such that $\mathrm{E}(u_g) = \delta_{g,e}$ for all $g \in \Gamma$.

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- there is a normal conditional expectation $\mathrm{E} \colon M \longrightarrow A$ such that $\mathrm{E}(u_g) = \delta_{g,e}$ for all $g \in \Gamma$.

Given a free, ergodic action $\Gamma \curvearrowright X$, we obtain a Cartan subalgebra $\mathrm{L}^\infty(X) \subseteq \mathrm{L}^\infty(X) \rtimes \Gamma$. It is of type II_1 if and only if the action is pmp.

Type II_1 equivalence relations

Definition

A type II_1 equivalence relation is an equivalence relation $\mathcal{R} \subseteq X \times X$ for a standard probability measure space (X, μ) such that

- $\mathcal{R}$ is a measurable subset of $X \times X$,
- orbits of $\mathcal{R}$ are countable,
- every partial automorphism of X whose graph lies in $\mathcal{R}$ is probability measure preserving
- $\mathcal{R}$ is ergodic, that is $\mathcal{R}$-saturated subsets are null or conull in X.

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Countability assumption imposed for measure theoretic reasons.

Equivalence relations and group actions

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The topological analogue is the effective quotient of a transformation groupoid.

The Feldman-Moore theorem

Let $\mathcal{R}$ be a II_1 equivalence relation on X. Then $\mathrm{L}(\mathcal{R})$ is the unique von Neumann algebra M which

- is generated by a copy of $\mathrm{L}^\infty(X) \subseteq M$ together with
- a multiplicative family of partial isometries $(u_\varphi)_{\varphi \in [[\mathcal{R}]]}$ satisfying $u_\varphi f = {}^\varphi f u_\varphi$ for all $\varphi \in [[\mathcal{R}]]$ and all $f \in \mathrm{L}^\infty(X)$, and
- there is a normal conditional expectation $\mathrm{E} \colon M \longrightarrow \mathrm{L}^\infty(X)$ satisfying $\mathrm{E}(u_\varphi) = \mathbb{1}_{\mathrm{Fix}(\varphi)}$.

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This construction can be made to accommodate a 2-cocycle $\sigma \in \mathrm{Z}^2(\mathcal{R}, \mathrm{S}^1)$ giving rise to a Cartan subalgebra $\mathrm{L}^\infty(X) \subseteq \mathrm{L}(\mathcal{R}, \sigma)$.

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Theorem (Feldman-Moore 1977)

Every Cartan subalgebra of a II_1 factor arises from a unique II_1-equivalence relation with a 2-cocycle.

Cartan subalgebras in the hyperfinite II_1 factor

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Corollary

The hyperfinite II_1 *factor has a unique Cartan subalgebra up to conjugacy by an automorphism.*

Unitary conjugacy of Cartan subalgebras

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Theorem (Speelman-Vaes 2012)

Let M be a separable II_1 factor. Then the set of Cartan subalgebras of M is a standard Borel space with the Effros Borel structure and the equivalence relation of unitary conjugacy is Borel.

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There is a separable II_1 factor M such that the space of Cartan subalgebras of M with the equivalence relation of being conjugate by an automorphism is complete analytic.

Popa's intertwining-by-bimodules

Theorem (Popa 2006)

Let M be a II_1 factor and $A, B \subseteq M$ two Cartan subalgebras. Then A and B are unitarily conjugate if and only if one of the following equivalent conditions is satisfied.

- *There is a homomorphism $\varphi\colon pA \longrightarrow qB$ and a partial isometry $v \in pMq$ such that $av = v\varphi(a)$ for all $a \in pA$.*
- *There is no net of unitaries $(u_i)_i$ in A such that $\|\mathrm{E}_B(mu_in)\|_2 \to 0$ for all $m, n \in M$.*

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So far, no analogue for intertwining-by-bimodules is available in the C^*-algebraic world.

Deformation/rigidity theory

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- rigidity of a $B \subseteq M$ forcing any such family to converge uniformly in 2-norm to the identity on the unit ball of B.

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Theorem (Ozawa-Popa 2010)

Let $\mathbb{F}_n \curvearrowright X$ be a free, profinite action of a non-abelian free group. Then $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$ has a unique Cartan subalgebra up to unitary conjugacy.

Group-measure space Cartan subalgebras

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Theorem (Popa-Vaes 2010)

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}) *_{\mathrm{UT}_2(\mathbb{Z})} \mathrm{PSL}_2(\mathbb{Z})$ *and* $\Gamma \curvearrowright X$ *any free, mixing, pmp action. Then* $\mathrm{L}^\infty(X) \rtimes \Gamma$ *has a unique group-measure space Cartan subalgebra. The action is even* W^**-superrigid.*

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Theorem (Ioana 2011)

The Bernoulli action of any property (T) is W-superrigid.*

From absence of Cartan subalgebras to uniqueness of Cartan subalgebras

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Strengthenings of absence of Cartan subalgebras for group von Neumann algebras are relativised to obtain uniqueness of Cartan subalgebras in group-meausure space constructions.

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If Γ is a hyperbolic group, then $\mathrm{L}\Gamma$ is strongly solid, that is $\mathrm{N}_M(A)''$ is amenable for every diffuse von Neumann subalgebra $A \subseteq \mathrm{L}\Gamma$. In particular, $\mathrm{L}\Gamma$ has no Cartan subalgebra.

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Theorem (Popa-Vaes 2014)

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Theorem (Chifan-Ioana-Osin-Sun 2021)

There are (concrete) examples of icc, property (T) groups that are W^*-superrigid. They are extensions $1 \to (\mathbb{Z}/2)^{\Lambda} \to \Gamma \to \Lambda \longrightarrow 1$ *for hyperbolic groups* Λ *with property (T).*

Uniqueness of Cartan subalgebras and ℓ^2-cohomology

Theorem (Popa-Vaes 2013)

Let Γ *be a weakly amenable group with positive first* ℓ^2*-Betti number. Then* $\mathrm{L}^\infty(X) \rtimes \Gamma$ *has a unique Cartan subalgebra up to unitary conjugacy for any free, ergodic, pmp action* $\Gamma \curvearrowright X$

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The group-measure space construction associated with any free,ergodic, pmp action of a group with some positive ℓ^2*-Betti number has a unique Cartan subalgebra up to unitary conjugacy.*

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A generalisation to arbitrary von Neumann algebras with a Cartan subalgebra could be thought of using Gaboriou's ℓ^2-Betti numbers for II_1 equivalence relations. But I have not heard of version in any conversion I recall.

Uniqueness of Cartan subalgebras and ℓ^2-cohomology

Theorem (Popa-Vaes 2013)

Let Γ be a weakly amenable group with positive first ℓ^2-Betti number. Then $\mathrm{L}^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra up to unitary conjugacy for any free, ergodic, pmp action $\Gamma \curvearrowright X$

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A positive solution would give computable, non-zero cohomological invariants for II_1 factors – a major open problem.

Beyond absence and uniqueness of Cartan subalgebras

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Theorem (Connes-Jones 1982)

There is a II_1 *factor* M *that has exactly two Cartan subalgebras up to conjugacy by an automorphism.*

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Problem

Which eqivalence relations can arise from unitary conjugacy/ conjugacy by an automorphism of Cartan subalgebras in a II_1 factor? In particular, for any $n \in \mathbb{N}$ is there a II_1 factor with exactly n Cartan subalgebras up to unitary conjugacy / conjugacy by an automorphism.

Thank you for your attention!