Non-Hausdorff groupoids and purely infinite C*-algebras

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Let G be topological groupoid.

- $G^{(0)} \subseteq G$ is locally compact and Hausdorff (in the relative topology)
- A set $B \subseteq G$ is a *bisection* if the restrictions $r|_B$ and $s|_B$ are injective.
- $\mathrm{Iso}(G) = \{\gamma \in G : r(\gamma) = s(\gamma)\}$

Definition

G is *étale* if r, s are local homeomorphisms.

Every étale groupoid G has a basis of open Hausdorff bisections.

An étale groupoid is *ample* if it has a basis of compact open bisections.

Examples

- Groupoids of Self-Similar actions (e.g. Grigorchuk Group)
- Groupoids of inverse semigroups

Non-Hausdorff groupoids introduce new wrinkles:

- $G^{(0)}$ is no longer closed. (Units can converge to elements outside $G^{(0)}$.)
- Compact sets are not necessarily closed.
- Products of continuous functions may not be continuous.

Fortunately the groupoid operations play nicely with the non-Hausdorff elements.

Definition

We say x and y in G *cannot be separated* if for all open sets $U, V \subseteq G$ with $x \in U$, $y \in V$ we have $U \cap V \neq \emptyset$. These are the *problem points* denoted $\mathcal{P}(G)$.

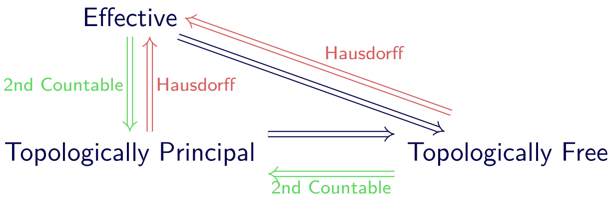
- If x and y cannot be separated then $r(x) = r(y)$ and $s(x) = s(y)$
- If $x \in \mathcal{P}(G)$ then $r(x)Gs(x) \subseteq \mathcal{P}(G)$.
- If K is compact in G then $r(K)$ and $s(K)$ are compact and closed in $G^{(0)}$ (w.r.t. relative topology.)

Definition

Let G be an étale groupoid. We say G is *minimal* if, for all $u \in G^{(0)}$ the orbit $r(s^{-1}(u))$ is dense in $G^{(0)}$.

Definition

1. G is *topologically principal* if the set of units with trivial isotropy is dense in $G^{(0)}$.
2. G is *effective* if $\mathrm{Int}(\mathrm{Iso}(G)) = G^{(0)}$.
3. G is *topologically free* if $\mathrm{Int}(\mathrm{Iso}(G) - G^{(0)}) = \emptyset$.



Definition

Let $C_c^0(G)$ be the set of functions $f : G \to \mathbb{C}$ such that there exists an open subset $V \subseteq G$ and:

1. V is Hausdorff
2. f vanishes outside V
3. $f|_V$ is continuous and compactly supported in V (i.e. $f \in C_c(V)$).

Let $\mathcal{C}_c(G)$ be the span of functions in $C_c^0(G)$. These functions are *not* in general continuous.

We define multiplication and adjoints on $\mathcal{C}_c(G)$ as follows:

$$(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}$$

For $u \in G^{(0)}$, we have left regular representations $L_u : \mathcal{C}_c(G) \to B(\ell^2(Gu))$ satisfying
$$L_u(f)\delta_\gamma = \sum_{\alpha \in Gr(\gamma)} f(\alpha)\delta_{\alpha\gamma} \text{ for } f \in \mathcal{C}_c(G).$$

Definition

For $f \in \mathcal{C}_c(G)$ we can define:
$$\|f\|_r := \sup\{\|L_u(f)\| : u \in G^{(0)}\}$$
The reduced C*-algebra $C_r^*(G)$ is the completion of $\mathcal{C}_c(G)$ under $\|\cdot\|_r$.

Let $j : C_r^*(G) \to B(G)$ be the Renault j-map.
$$J_{\mathrm{sing}} := \{a \in C_r^*(G) : \mathrm{supp}(j(a)) \text{ is meagre}\}$$
J_{sing} doesn't intersect $C_0(G^{(0)})$ and so obstructs both simplicity and to extending faithful maps.

For a Hausdorff groupoid G restricting $f \in C_c(G)$, $f|_{G^{(0)}}$ extends to a conditional expectation $E : C_r^*(G) \to C_0(G^{(0)})$.

Let G be a non-Hausdorff groupoid and let $f \in \mathcal{C}_c(G)$. The restriction $f|_{G^{(0)}}$ has an issue: $f|_{G^{(0)}}$ might not be in $C_0(G^{(0)})$ or even in $\mathcal{C}_c(G)$. We have two options:

Definition

The restriction $f|_{G^{(0)}}$ extends to a faithful expectation on bounded Borel functions

$$E_r : C_r^*(G) \to \mathfrak{B}(G^{(0)}) \quad (\subseteq C_0(G^{(0)})^{**})$$

Or we can take a quotient to kill singular functions:

Proposition

Let

$$\mathfrak{M}(G^{(0)}) := \{g \in \mathfrak{B}(G^{(0)}) : \mathrm{supp}(g) \text{ is meagre}\}$$

The restriction $f|G^{(0)} \in \mathfrak{B}(G^{(0)})/\mathfrak{M}(G^{(0)})$ *extends to an expectation*

$$E_{\mathrm{ess}} : C_r^*(G) \to \mathfrak{B}(G^{(0)})/\mathfrak{M}(G^{(0)}) \quad (\subseteq \mathcal{M}_{\mathrm{loc}}(C_0(G^{(0)})))$$

Definition

The essential groupoid C*-algebra is the quotient

$$C_{\mathrm{ess}}^*(G) := C_r^*(G)/\{a \in C_r^*(G) : E_{\mathrm{ess}}(a^*a) = 0\}$$

E_{ess} is faithful on $C_{\mathrm{ess}}^*(G)$ and $C_0(G^{(0)})$ has the i.i.p. in $C_{\mathrm{ess}}^*(G)$.

Definition (cf Bönicke-Li, Rainone-Sims and Ma)

Let G be an ample non-Hausdorff groupoid with compact unit space. Consider $C_c(G^{(0)},\mathbb{Z})^+ = \{\sum 1_U : U \subseteq G^{(0)},\ \text{compact open}\}$. For $f,g \in C_c(G^{(0)},\mathbb{Z})$ we say $f \leq_G g$ if there exist finite bisections B_i such that

$$f \leq \sum_{i=1}^{n} 1_{s(B_i)} \text{ and } \sum_{i=1}^{n} 1_{r(B_i)} \leq g$$

We say $f \sim_G g$ if $f \leq_G g$ and $g \leq_G f$.

The quotient $C_c(G^{(0)},\mathbb{Z})^+/\sim_G$ gives a monoid with addition and partial order

$$[f]_G + [g]_G = [f+g]_G \qquad [f]_G \leq [g]_G \text{ if } f \leq_G g$$

We call this monoid the *type semigroup of* G, labeled $\mathrm{type}(G)$.

Following Bönicke-Li, Rainone-Sims and Ma we define a notion of "paradoxicality" for G.

Definition (cf Bönicke-Li, Rainone-Sims and Ma)

- For integers $j > k > 0$ we say $f \in C_c(G^{(0)}, \mathbb{Z})^+$ is (j,k)-*paradoxical* if $jf \leq_G kf$ i.e. there exist bisections $B_i \subseteq G$

$$jf \leq \sum_{i=1}^{n} 1_{s(B_i)} \text{ and } \sum_{i=1}^{n} 1_{r(B_i)} \leq kf$$

- f is *completely non paradoxical* if it is not (j,k)-paradoxical for any $j > k > 0$.
- G is *completely non-paradoxical* if every $f \in C_c(G^{(0)}, \mathbb{Z})^+$ is completely non-paradoxical.

Theorem (ACaHMT)

Let G be an ample, minimal, topologically free groupoid with compact unit space. Consider the following:

1. *Every $f \in C_c(G^{(0)}, \mathbb{Z})^+$ is $(2,1)$-paradoxical.*
2. *$C^*_{\mathrm{ess}}(G)$ is purely infinite simple.*

Then (1) $\implies$ (2). If $\mathrm{type}(G)$ is almost unperforated then (2) $\implies$ (1) and we have equivalence.

Theorem (ACaHMT)

Let G be an ample, minimal groupoid with compact unit space. The following are equivalent:

1. *$C^*_{\mathrm{ess}}(G)$ is stably finite*
2. *G is completely non-paradoxical*

Theorem (ACaHMT)

Let G be an ample, minimal, topologically free groupoid with compact unit space. Suppose that $\mathrm{type}(G)$ is almost unperforated.

Then $C^*_{\mathrm{ess}}(G)$ is simple and either purely infinite or stably finite.