CMO Workshop: Quantum Markov Semigroups and Channels: Special Classes and Applications 24w5240

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August 18 – 23, 2024.

Topics of the workshop

- Generalized gaussianity and gaussian states
- Quantum graph homomorphisms and quantum walks on graphs
- Quantum Markov Semigroups channels and classes: Gaussian, Low-Density Limit, Weak Coupling Limit, G-circulant
- Quantum Markov Semigroups analysis: decoherence-free subalgebras, invariant states, spectral gap, operator space fragmentation, population inversion and Ricci curvature bounds
- Quantum probability: quantum CLT, quantum entropy and quantum Steins Lemma
- Quantum fields: Control and Supersymmetry

1 Overview of the field

Quantum channels and semigroups of completely positive, identity preserving maps on a von Neumann algebra, called quantum Markov semigroups (QMS) (or dynamical semigroups in the Physics language) have become the key structure for describing open system dynamics in all of Physics. Their importance for transmission of information and description of irreversible processes involving dissipation and decoherence can hardly be underestimated, and with the recent impetus from quantum technologies, where noise is an unavoidable challenge, their role has been steadily growing. Even though much is known in the case of classical channnels and Markov semigroups on commutative algebras, our structural understanding of the quantum non-commutative case is still very limited. One of the main reasons being the high dimensionality and the high level of generality involved (for instance diffusion and jump behaviour, both can coexist in the same quantum stochastic process). Therefore the study study in detail of special classes of channels and semigroups that are physically relevant and with a rich structure has increased in the recent years.

The outcome of this workshop provides fertile ground for a more profound comprehension of the non-commutative mathematics involved, as well as new findings on mathematical models for open quantum systems. It brought together researchers from various groups based in different countries around the globe, all focused on related research areas, which lie at the intersection of functional analysis, quantum mechanics, probability, and operator theory. Despite their diverse perspectives and objectives, these groups utilize closely related techniques.

2 Recent Developments and Open Problems

Gaussian Quantum Markov Semigroups

Gaussian (quasi-free) QMS describe the evolution of open quantum systems of bosons interaction with the surrounding environment, they generalize bosonic quadratic Hamiltonians. They can be defined either through their explicit action on Weyl operators or through their generator. It has been proven that the Kraus operators and hamiltonian present in GKSL generator of such semigroups need to have the following structure

$$\begin{aligned} H &= \sum_{j,k=1}^{d} \left(\Omega_{jk} a_j^\dagger a_k + \frac{\kappa_{jk}}{2} a_j^\dagger a_k^\dagger + \frac{\overline{\kappa_{jk}}}{2} a_j a_k \right) + \sum_{j=1}^{d} \left(\frac{\zeta_j}{2} a_j^\dagger + \frac{\overline{\zeta_j}}{2} a_j \right), \\ L_\ell &= \sum_{k=1}^{d} \left(\bar{v}_{\ell k} a_k + u_{\ell k} a_k^\dagger \right), \end{aligned}$$

where $1 \le m \le 2d$, $\Omega := (\Omega_{jk})_{1\le j,k\le d} = \Omega^*$ and $\kappa := (\kappa_{jk})_{1\le j,k\le d} = \kappa^T \in M_d(\mathbb{C})$, are $d \times d$ complex matrices with Ω Hermitian and κ symmetric, $V = (v_{\ell k})_{1\le \ell\le m, 1\le k\le d}$ and $U = (u_{\ell k})_{1\le \ell\le m, 1\le k\le d} \in M_{m\times d}(\mathbb{C})$ are $m \times d$ matrices and $\zeta = (\zeta_j)_{1\le j\le d} \in \mathbb{C}^d$.

This class of semigroups is characterized by the following formula for the action of a Gaussian semigroup on Weyl operators $W(z)$.

$$\mathcal{T}_t(W(z)) = \exp\left(-\frac{1}{2} \int_0^t \Re\langle e^{sZ} z, C e^{sZ} z\rangle \, ds + i \int_0^t \Re\langle \zeta, e^{sZ} z\rangle \, ds \right) W(e^{tZ} z),$$

where the real linear operators Z, C on $\mathbb{C}^d$ are

$$\begin{aligned} Zz &= \left[\left(U^T \overline{U} - V^T \overline{V} \right)/2 + i\Omega \right] z + \left[\left(U^T V - V^T U \right)/2 + i\kappa \right] \overline{z}, \\ Cz &= \left(U^T \overline{U} + V^T \overline{V} \right) z + \left(U^T V + V^T U \right) \overline{z}. \end{aligned}$$

It turns out that the stability of the operator Z is sufficient for the existence of a unique gaussian invariant state ρ, even more if $C_Z = C - i(Z^* J + JZ) > 0$ then it is faithful.

In this case it is seen that, considering the two well known embeddings, the GNS embedding and the KMS embedding, lead to different notions of spectral gap. In any case this quantity gives a bound for the exponential decay of the evolution. It was showed that under the above assumptions the spectral gap exists and was computed explicit and that such conditions are not restrictive. The relationship between the different notions of spectral gap is an open problem.

In a different direction the decoherence-free subalgebras of Gaussian QMS can be characterized as generalized commutants of linear combinations of the itterated commutators of the form $\delta_H^n(L_l), \delta_H^n(L_l^*)$ where $\delta_H(x) = [H, x]$. In this case the decoherence-free subalgebra is given by the tensor product of L^∞ and a type I factor.

Several questions arise, for instance when decoherence takes place and what is the connection between the Weak Coupling Limit Type semigroups and gaussian semigroups. Some discussions towards this end took place during the workshop.

Infinite mode gaussian states (Boson case)

Quantum Gaussian states on Boson Fock spaces serve as the quantum analogs of Gaussian distributions. There is a substantial body of literature on quasifree states on CCR (Canonical Commutation Relation) algebras, commonly referred to as quantum Gaussian states or squeezed states, in-depth discussions and further references can be found in several monographs [5, 4, 20]. Recently, finite-mode Gaussian states have gained increasing attention due to their significance in quantum information theory.

Gaussian states $\rho \in L_1(\mathcal{H})$ in infinite modes are defined as states whose quantum characteristic function or non-commutative Fourier transform is gaussian (up to normalization), i.e.,

$$\mathcal{F}[\rho](z) = \text{tr}(\rho W_z) = \frac{1}{\sqrt{\pi}} e^{-i(w,z)-\frac{1}{2}(z,Sz)} \quad z \in \ell_2(\mathbb{N})$$

where $W_z,\ z \in \ell_2(\mathbb{N})$ is the Weyl operator on the stabilized Fock space representaion of the Canonical Commutations Relations (CCR's). Different approaches have been recently developed. One is the use of a refinement of Shale operators and Bogoliubov transformations together with an extension of Williamson's normal form to infinite dimensions which provides explicit description for every quantum Gaussian state in terms of symplectic eigenvalues of the covariance operator. Another one uses Yosida's aproximations to deduce a recurrence relation that a gaussian state must satisfy which is reminiscent of the classical setting, with this a notion of integrability of unbounded observables can be defined.

G-circulant Quantum Markov Semigroups

Circulant QMS were first introduced a decade ago. Their rich structure, stemming from the numerous symmetries of the underlying group $\mathbb{Z}_n$, has enabled explicit computations of invariant states, entropy production rate. . The well-known spectral properties of circulant matrices extend to circulant GKSL generators, allowing for the derivation of bounds on the spectral gap. A natural next step is to generalize these results to arbitrary finite groups.

Given a finite group G, the GKSL generator of a G-circulant QMS is of the form

$$\mathcal{L}(x) = i[H,x] + \mathcal{L}_0(x) = i[H,x] + \sum_{g\in G} \alpha_g U_g^* x U_g$$

where U_g is the unitary left regular representation acting on $\ell_2(G)$. Although the structure of invariant states and long time behaviour is completely known for $\mathcal{L}_0$, an estimate for the spectral gap and entropy production rate for non-tracial invariant states is not yet determined. On the other hand, when interesting hamiltonians are considered such as cyclic or diagonal non trivial hamiltonians, the problems stated above remain relevant. A characterization of these generators in terms of the invariance of the G-circulant subspaces and some extra conditions should be attainable in the near future. Possible generalizations of this G-circulant structure to infinite discrete groups remain open.

3 Presentation Highlights

The ubiquity of gaussinity in different areas of mathematics was one of the central topics in our workshop. By considering the cannonical decomposition of a random variable X in the context of interacting Fock spaces (IFS)

$$X = a^+ + a^0 + a^-,$$

Luigi Accardi showed how the classical moments may be obtained by quantum covariances in the so called process of gaussianization

$$E(X^n) = \langle \Phi_0, X^n \Phi_0 \rangle = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\in\{+,0,-\}}^{n} \langle \Phi_0, a^{\varepsilon_1} \cdots, a^{\varepsilon_n} \Phi_0 \rangle$$

where Φ_0 denotes the vacuum vector and a^+, a^-, a^0 are creation, anihiliation and preservation operators acting on gradation spaces of the IFS. The above quantities are obtained via the *quantum moments* of the random variable X

$$\langle \Phi_0, a^{\varepsilon_1} \cdots, a^{\varepsilon_n} \Phi_0 \rangle.$$

A wide perspective on the recent developments in the study of Gaussian QMS ranging from the structure of the decoherence-free subalgebras, the existence of invariant states to the computation of the spectral gap. Gaussian QMS are defined and semigrups with preserve gaussian states. **Franco Fagnola** introduced the gaussian GKSL generators and their structure in terms of a quadratic hamiltonian and Kraus operators as linear combinations of creation and anihilation. By the minimal semigroup method, properties of the Markovian dynamics such as irreducibility, ergodicity, existence of invariant states were studied. **Federico Girotti** showed some results characterizing the set of invariant states, the restrictions on the generator imposed by the existence of an invariant state and what consequences it has on the dynamics (convergence to equilibrium and environment-induced decoherence). In the case where a faithful invariant state exists, **Emmanuela Sasso** computed explicitly the spectral gap (the optimal exponential convergence rate) as the the exact the lowest eigenvalue of a certain matrix determined by the diffusion and drift matrices in the GNS and KMS embeddings of the semigroup when the hamiltonian part was not considered. **Damiano Poletti** presented a structure result for the generator of GNS-symmetric gaussian semigroups which showed the amount of degrees of freedom available when constructing such QMSs. In a related direction, **Roberto Quezada** presented some results on the conditions for a WCLT QMS to preserve gaussian states, i.e., to be Gaussian QMS, in the d-dimensional Fock representations of the CCR's. A sufficient condition, for Gaussian quantum Markov semigroup with quadratic hamiltonian in $a, a^\dagger$, is that the jump operators must be either multiples of a or $a^\dagger$ to be weak-coupling limit type (WCLT).

Veronica Umanità showed how the structure of the decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ in the atomic case induces a decomposition of the system into its noiseless part and purely dissipative part. This decomposition is helpful to determine the structure of invariant states among other things. When there is a faithful invariant state, the decoherence-free subalgebra must be atomic and decoherence takes place.

A construction of Block Modified Random Matrices was shown to be free when decomposing it with respecto to a circulant-pattern by **Octavio Arizmendi**.

The so-called relaxation rates provide important characteristics both for classical and quantum processes since they control how fast the system thermalizes, equilibrates, decoheres, and/or dissipate. In this context, **Dariusz Chruscinski** proved the conjecture formulated few years ago that any quantum dynamical semigroup implies that a maximal rate is bounded from above by the sum of all relaxation rates divided by the dimension of the Hilbert space.

George Androulakis presented a walkthrpugh to the proof of inequalities leading to the proof of the generalized quantum Stein's. This lemma is a fundamental result in quantum information theory that extends classical hypothesis testing to the quantum domain. Specifically, the lemma concerns distinguishing between two quantum states ρ and σ based on n copies of the state. Its generalization provides a powerful tool for understanding the efficiency of state discrimination in quantum information theory, connecting it to fundamental quantities like quantum relative entropy and resource measures.

Tiju Cherian John gave an overview bridging classical and quantum realms of information theory, focusing on entropy and central limit theorems. Highlighting Lieb's 1978 conjecture on the monotonicity of differential entropy and its 2004 proof by Artstein, Ball, Barthe, and Naor, which tied entropy power inequalities to the theorem in the classical setting. Transitioning to quantum theory, it covered the quantum central limit theorem by Cushen and Hudson (1971) and the unresolved issue of von Neumann entropy monotonicity. The talk concluded by discussing new results on the entropy of the distribution of observables, with implications for quantum information theory.

Priyanga Ganesan presented an introduction to quantum graphs arising in the context of operator systems. These are generalizations of classical graphs which serve the role of *confusability graphs* associated with quantum channels.

Quantum random walks and the phenomena of localization was explored by **Tulio Gaxiola** via the method of quantum decomposition for spectral analysis of graphs for the a specific type of infinite graphs called spidernets.

One extension of stochastic analysis, control and nonlinear filtering of classical fields to quantum nonlinear evolutions, such as Yang-Mills-Higgs-Spinor systems, and operator theoretic evolutions, such as Hudson-Parthasarathy and GKLS evolutions, was presented by **Sivaguru Sritharan**.

A super Fock space of a disjoint union of super Hilbert spaces which is equivalent to super tensoring of boson (even) part symmetrically and that of fermion (odd) part antisymmetrically of the super particle Hilbert space was showed, by **Radhakrishnan Balu**, to lead to a super Fock space that is a disjoint union of bosonic and fermionic spaces, i.e., is $\mathbb{Z}_2$-graded.

Alexander Teretenkov presented the phenomena of operator space fragmentation for a quantum Markov semigroup for a many-body system where one has at least exponentially (in particle number) many operator subspaces that are invariant with respect to the semigroup and these subspaces have linear (in particle number) dimension and trivial intersection. Some dissipative models that exhibit operator space fragmentation are Jordan products of so-called Onsager strings.

Aurel Stan described the operators anihilation, creation and preservation present in the the cannonical quantum decomposition of random variables for the Meixner class random variables.

In the context of a quantum system driven by boundary conditions **Eric Carlen** presented a study of the stationay states focusing on the structure, the unicity and the speed of relaxation towards equilibrium. In the final section of his presentation, he addressed the problem of hyporcoercivity in this setting.

Melchior Wirth and Haonan Zhang discussed a notion of Ricci curvature lower bound for for quantum Markov semigroups which uses an intertwining technique that leads to some log Sobolev functional inequalities. They showed applications to quantum Ornstein-Uhlenbeck semigroups and depolarizing semigroups QMSs.

The boundedness of Gaussian Riesz potentials I_β for $\beta \geq 1$ on regular enough Gaussian variable Lebesgue spaces was presented by **Wilfredo Urbina**, in the last part of his talk he gave a brief introduction to his book "Gaussian Harmonic Analysis" highlighting the key results as asked by the organizers with the aim of illustrating them to indicate future directions of investigation in quantum stochastic Gaussian analysis.

Fernando Guerrero Poblete and Marco Antonio Cruz de la Rosa offered state of the art research on the study of WCLT generators. The former showed some conditions of a 3-level non-generic WCLT semigroup in order to exhibit the population inversion phenomea. The later studied the special class of uniform and completely non equilibrium invariant states by considering the k-jump interaction graph $G_{\mathcal{L}}$ associated with the system hamiltonian.

Josué Ivan Rios Cangas presented an original framework to rigorously define the moments of a normal unbounded observable in a gaussian state by using Yosida approximation to overcome the problem of tracing a general unbounded operator. This approach allows the deduction of well known formulae for the mean value vector and the covariance matrix of a Gaussian state. This notion is used to characterize Gaussian states in terms of the moments of the field operator.

G-circulant QMS is the generalization of circulant QMS, where the underlying group is $\mathbb{Z}_n$, to the case of an arbitrary non commutative group G. Using G-circulant invariant subspaces, **Josué Vázquez-Becerra** showed a finer decomposition of the action of the generator which was, in a sense, a decomposition in "irreducible components". The structure of invariant states together with some algebraic properties were discussed. In the last part, diagonal non G-circulant hamiltonian perturbations of the generator were considered.

Special introductory sessions led by early-career researchers provided PhD students attending the workshop with an opportunity to learn about advanced topics and gain valuable context for better understanding the primary subjects addressed in the subsequent talks:

1. **Tiju Cherian John** - An introduction to Quantum Probability
2. **Damiano Poletti** - An introduction to Gaussian QMS
3. **Federico Girotti** - A brief introduction to ergodic theory in quantum Markov dynamics
4. **Marco Cruz de la Rosa** - An introduction to WCLT generators
5. **Fernando Guerrero** - An introduction to Low-Density generators

Conversely, during the PhD Students Session, young researchers presented some of the challenges they are currently investigating as part of their doctoral programs and received insightful feedback from experienced scholars:

1. **Saylé Sigarreta Ricardo** - Energy change after fusion
2. **María Guadalupe Salgado Castorena** - Low dentisy limit QMS: Beyond the two-generic case
3. **Luis Daniel Regalado Hernández** - Van Hove Quadratic Hamiltonians

4 Scientific Progress Made

This workshop fostered formation of new collaborations between different and diverse groups of researchers and the continuation of the existing ones. It provided an ideal environment for exchanging ideas, sharing ongoing work, and identifying potential areas for joint research sparking new connections and collaborations that could lead to future advancements in their respective areas of study.

- Some existing collaborations that were initiated prior to the workshop were further developed. Notably, research on the asymptotic behavior of Quantum Markov Semigroups (QMS) with specific classes of generators saw significant progress. This work is crucial for understanding long-term quantum system dynamics and stability.
- A structural result for the generator of GNS-symmetric Gaussian QMSs was presented, revealing the degrees of freedom available when constructing such semigroups.
- Significant strides were made in the understanding of intersections between the Weak-Coupling Limit (WCLT) and Gaussian semigroups and conditions characterizing these generators.
- Integrability of observables with respect to gaussian states was another area of discussion. Participants examined novel approaches to known results regarding the covariances in the covariance of a gaussian state.
- Advances were made in the study of perturbed G-circulant semigroups. Theoretical developments regarding their spectral properties were discussed, and there were fruitful exchanges on how these findings could be applied to analyze quantum channels and noise processes. Some ideas to extend this class to infinite groups were addressed.
- The self-adjointness of quadratic Van Hove Hamiltonians, particularly in cases where the annihilation operator is not closable, was explore. This topic has critical implications for quantum field theory and quantum statistical mechanics, providing new insights into the spectral theory of unbounded operators.
- A concrete example concerning the 3-generic case of a GKSL generator of Low-density was discussed. Researchers shared recent advancements in this area, including characterizations of diagonal invariant states.

The workshop featured a dynamic exchange of ideas, with engaging discussions occurring not only during the presentations but also throughout breaks and mealtimes. Moreover, the special sessions towards young researchers and phD students held on Monday and Tuesday evening, provided the participants with dedicated opportunity to discuss specific challenges in their carreer and phD programs.

5 Outcome of the Meeting

This workshop presented an invaluable opportunity for different groups from around the world to establish dialogue, share ideas and open problems which will stimulate future collaborations.

Quantum Markov Semigroups (QMSs) and Quantum Probability, in general, are foundational elements within emerging theoretical frameworks where they offer significant contributions. Many challenges in this field arise from specific issues in Physics or Quantum Information. The study of the different classes of channels is crucial to the discovery of meaningful and practically applicable results.

We know from e-mail contacts we had after the meeting that, in particular:

- Some participants (Luigi Accardi, Tulio Gaxiola, Josué Rios-Cangas, Roberto Quezada, Josué Vázquez, Jorge Bolaños-Servín, Fernando Guerrero, Guadalupe Salgado, Luis Daniel Regalado) will attend a mini-workshop named “Gaussianity: Generalizations and Applications” in UAM-Azcapotzalco (Mexico City) some time after the workshop.
- A possible visit of Franco Fagnola to UAM-I (Mexico City) was discussed in order to collaborate with Roberto Quezada and Jorge Bolaños-Servín on the intersection of gaussian and WCLT GKSL generators.

- Emanuela Sasso, Veronica Umanità and Priyanga Ganesan (and possibly Wilfredo Urbina) will start a collaboration proyect to apply quantum graph and operator system methods in the study of Gaussian QMSs and Weyl algebras on a group, i.e. where Weyl operators $W(g)$, g in the group with a symplectic form. This activity also intersects with that of the group Women in Operator Algebra workshops.

- Talks to develop a phD program option for Enedino Onofre in UAM-I once his masters program in UAGro is competed were discussed.

- Jorge Bolaños-Servín and Octavio Arizmendi will start a collaboration to generalize the circulant results in free probability to the G-circulant case.

- An application of quantum graphs and operator systems to study the confusability graphs associated with G-circulant completely positive maps will be discussed by Jorge Bolaños-Servín and Priyanga Ganesan.

- A workshop to be held in Mexico City in October with the theme of Theory of operators, Graphs and Applications (TOGA) started to be organized by Josué Rios-Cangas, Tulio Gaxiola and Jorge Bolaños-Servín.

- Priyanga Ganesan is plannig a visit to collaborate with European participants in 2025.

- Similar themed workshops during 2025 are being planned.

Understanding the current state of research and the open problems will serve as valuable guidance for researchers. There is strong reason to believe that the methods and ideas discussed during the workshop, along with new collaborations, will ignite further investigations and lead to new discoveries in the coming years.

PhD students who attended the workshop likely benefited in several key ways such as:

- Exposure to Advanced Topics: They were introduced to cutting-edge research areas within Quantum Markov Semigroups, Quantum Probability, and related fields, which are foundational to emerging theoretical frameworks in Physics and Quantum Information. This exposure is invaluable for broadening their understanding and for integrating these concepts into their own research.

- Opportunities for Collaboration: The workshop provided a platform for young researchers to network with established experts and peers from around the world. This can lead to potential collaborations, joint projects, and shared research initiatives in the future such as Franco Fagnola's visit to UAM-I in January and Enedino Onofre's plan for a phD proyect.

- Constructive Feedback: By presenting their current research problems in the phD problems sessions, PhD students received constructive feedback and insights from experienced researchers. This will help them refine their approaches, methodologies, and potentially rethink certain aspects of their work.

- Guidance on Open Problems: The workshop highlighted the current state of research and ongoing challenges in the field. This guidance could help PhD students identify significant problems to focus on in their own studies, shaping their research trajectory towards impactful outcomes.

- Skill Development: Engaging in discussions, presenting research, and participating in collaborative sessions would have enhanced their academic and professional skills, including communication, critical thinking, and problem-solving abilities.

Overall, the workshop can be a pivotal experience for PhD students, providing them with a clearer perspective on their field, new ideas for their research, and valuable connections for their academic careers.

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