Recent advances in Comparison Geometry

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1 Overview of the Field

The fundamental idea of comparison geometry is that of trying to understand the shape of some geometric objects in terms of the shape of suitable model configurations, which are in general simpler objects, sometimes explicit, equipped with a remarkable number of symmetries. Typical outcomes of this profound idea are geometric inequalities. Among the most basic and earliest examples there certainly is the classical Isoperimetric Inequality, where a suitable scaling invariant area-to-volume ratio is compared with that of the ball in the Euclidean space (model situation).

In the last century, this mythological inequality has known several extensions and generalisations. It also possesses a relativistic counterpart known as the Riemannian Penrose Inequality where, roughly speaking, the mass-to-area ratio of a Black Hole is compared with that of the Schwarzschild solution. This is not surprising after all: General Relativity has in fact been defined as a geometric theory of gravity. As such, it is a big source of inspiration of deep geometric problems. Viceversa, deep advances in geometry provides fruitful insights in our understanding of the physical universe.

As it will become clearer with the description of the talks, the fields of Mathematical Analysis and Geometry that are tied with comparison geometry are the most diverse.

2 Recent Developments and Open Problems

It is quite difficult to give an account of the impressive advancements obtained in recent years that are connected with comparison geometry. We limit ourselves to mention a few of the most recent, that are to our advice particularly close to some of the topics discussed in the workshop, but that will not directly appear in the presentations summarized in Section 3. Moreover, they testifies the great advancements this field has been witnessing in the last few years.

The longstanding conjecture known as Milnor's conjecture, asserting that manifolds with nonnegative Ricci curvature always have a finitely generated first fundamental group, has prove to be false in high dimensions by Bruè, Naber and Semola [4].

A sharp, quantitive version of the Positive Mass Theorem, as conjectured by Huisken-Ilmanen [6], has been proved by Dong and Song [5].

On the curvature flows side, the longstanding Multiplicity One Conjecture has been proved by Bamler and Kleiner for the mean curvature flow of surfaces in the Euclidean 3-space [3].

Among the many open problems in the field, we report on one that connects many of the talks proposed. It concerns the concept of scalar curvature for nonsmooth Riemannian manifolds. Although a number of proposals have been given, a complete understanding is still lacking. This partly motivates the endeavor in stability estimates for geometric inequalities, the study of nonsmooth structures,new routes in comparison geometry based on curvature flows and PDEs techniques, and so on.

3 Presentations

We give a brief description of the presentations given. The references can be found in the slides or in the the videos made available in the official webpage of the workshop.

1. Yuguang Shi discussed conditions on noncompact manifolds preventing the existence of metrics with suitable lower bounds on the scalar curvature. Such conditions are related to a classical theorem of Llarull, asserting that a smooth non-round metric g on the sphere cannot have scalar curvature $R \geq n(n-1)$ and simultaneously satisfy $g \geq \overline{g}$, where $\overline{g}$ is the round metric of the n-dimensional sphere. In fact, Prof. Shi proposed a new Llarull-type theorem for noncompact spin manifolds.

2. Luciano Mari discussed the minimal graph equation on manifolds (M, g) with nonnegative sectional or nonnegative Ricci curvature. In the former case, he showed that if an entire solution u satisfies

$$u(x) = O(r(x)),$$

   where $r(x)$ is the distance between x and a fixed origin o, then (M, g) splits a cylinder, and u is accordingly affine. In the nonnegative Ricci curvature case, he showed that positive entire solutions are in fact constant. These results rest on a new, global gradient estimate.

3. Luca Benatti presented a number of results connecting theory and geometric applications of nonlinear potential theory and Inverse Mean Curvature Flow. Among the various results, he discussed the recovering of Huisken-Ilmanen's Georch monotonicity through a p-harmonic approximation, complemented with a geometric measure study of the level sets of p-harmonics, Penrose inequalities for isoperimetric and p-isocapacitary masses, as well as relations among such notions of mass and the more classical ADM mass.

4. Sven Hirsch presented several new stability and rigidity results for scalar curvature. In particular, he proved stability of Llarull's theorem in all dimensions using spin geometry. Additionally, he discussed some related questions which are motivated by General Relativity.

5. Stefano Borghini discussed new comparison geometry results in the context of substatic Riemannian manifolds. These can be viewed as generalizations of manifolds with nonnegative Ricci curvature, that can moreover be naturally endowed with a minimal compact boundary. They naturally emerge from static spacetimes satisfying the Einstein equations.

6. Kai Xu provided counterexamples to a conjecture of Gromov-Sormani, asserting that limits of 3-toris with scalar curvature that tends to zero, uniform upper bounds on volumes and diameters, and a positive lower bound on the area of closed minimal surfaces, are only flat tories. The counterexamples are called drawstrings, and in fact the limit spaces are flat out of a closed curve, where the metric degenerates.

7. Ovidiu Munteanu provided new comparison results mostly involving the scalar curvature only. They are based on suitable monotonicity formulas holding along the level sets of harmonic functions. Among the various application, they allowed to prove the 3D version of a conjecture of Gromov.

8. Nicola Gigli discussed new insights on Sobolev spaces and elliptic operators on Lorentzian spaces, with a systematic comparison with the Riemannian case. As possible applications of such new theory, he illustrated a very simple proof of the Galloway's Lorentzian splitting theorem.

9. Man-Chun Lee illustrated a number of results including Positive Mass Theorems, Llarull'type theorems, and torus rigidity theorems for C^0 metrics with singularities contained in a set of suitably high codimension, with scalar curvature bounds to be understood in the sense of approximation. The tool allowing to prove such results are in fact delicate smoothing procedures through the Ricci flow.

10. Melanie Graf discussed a number of comparison results in a Lorentzian setting, including mean curvature comparisons, area comparisons and Bonnet-Myers'type theorems, in this setting to be understood as incompleteness/singularity theorems.

11. Florian Johne considered comparison results for m-intermediate curvatures, extending the theorems of Bonnet-Myers and of Schoen-Yau to versions involving such curvatures.

12. Alessandra Pluda provided an overview on the theory of network flows. Among the new results she mentioned, we find an existence theorem past singularities and stability results for the steady case of networks with triple junctions and straight segments.

13. Christina Sormani considered the important conjecture of Gromov about about limits of manifolds with nonnegative scalar curvature: they should have some notion of nonnegative scalar curvature as well. Among the various results presented, she presented a special case of such conjecture that has been proved in the positive, where the limit is in fact a $W^{1,p}$-limit with nonnegative distributional scalar curvature. She also discussed many open questions in the field.

14. Jintian Zhu provided a new version of the Penrose inequality, generalizing the known one by Bray and Huisken-Ilmanen. It replaces the lower bound given in terms of the area of the horizon with a lower bound in terms of the systole, defined as the infimum of the area of two sided surfaces. In particular, it applies to extreme cases where classical Riemannian Penrose inequality does not apply.

15. Gioacchino Antonelli discussed recent advances obtained in the study of the isoperimetric problem on possibly nonsmooth noncompact manifolds with curvature lower bounds, including regularity and sharp estimates on the isoperimetric profile, as well as uniqueness of isoperimetric sets. He also highlighted the relation with a conjecture of Huisken about the positivity of the isoperimetric mass in the nonsmooth setting.

16. Martin Reiris showed, using techniques in comparison geometry,that there are vacuum static 3+1 black hole solutions, metrically complete but with a non-standard spatial topology, that cannot be put into rotation, that is, there are no non-static stationary metrics close to them.

17. Marcus Khuri presented the first examples of formally asymptotically flat black hole solutions with horizons of general lens space topology. As a by-product, he got new examples of regular gravitational instantons in higher dimensions.

18. Guofang Wang proposed a higher order generalization of the classical scalar curvature, itself supporting a generalization of Gauss-Bonnet Theorem. In connection with that, he introduced related notion of ADM mass, and presented some positive mass theorems in the asymptotically flat setting. He proposed some open problems in this field, and clarified the connection with Alexandrov-Fenchel inequalities.

19. José Espinar showed a Serrin-type Theorem for an eigenvalue equation for the Laplacian with Dirichlet boundary values in ring-shaped domains of the 2-dimensional sphere. This is the spherical analogue of a result of Agostiniani-Borghini-Mazzieri, itself discussed in Mazzieri's talk. As an application, he proved the a version of the critical catenoid conjecture under the additional assumption that its support function has infinitely many critical points.

20. Lorenzo Mazzieri presented Serrin-type theorems for the torsion equation in domains with disconnected boundaries, under the additional assumption that the set of maxima of the solution is infinite. The technique rests on comparison geometry arguments. He also showed a version of such result in the classification problem of static metrics with positive cosmological constant.

21. Jie Wu discussed Alexandrov-Fenchel inequalities involving weighted curvature integrals. In general, the proofs are based on suitable geometric evolution equations, typically the Inverse Mean Curvature Flow and variants. The geometric assumptions on the domain are related to convexity and starshapedness, allowing the flow to evolve smoothly forever.

22. Paula Burkhardt-Guim exposed a notion of ADM mass that is well defined on asymptotically flat metrics that are only C^0, and with nonnegative scalar curvature in a suitable sense, based on the properties of the Ricci flow issuing from them. The natural question of its nonnegativity naturally arose, as well as the relation with other notions of mass that are well defined in the same setting.

23. David Wiygul presented an asymptotic lower bound for the Bartnik mass of spheres with data close to that of the standard sphere in Euclidean space. The estimate relied on the construction of hyperannular fill-ins (spherical shells) which are approximately static vacuum.

24. Chao Xia presented a family of monotonicity formulas for p-harmonic functions holding in the setting of 3-manifolds with nonnegative scalar curvature. They encompass most of the monotonicity formulas already available in the field, and their constancy case in fact characterizes the Schwarzschild 3-space.

4 Outcome of the Meeting

As the diversity of topics discussed in this workshop testifies, a number of mathematical theories and problems are intertwined with comparison geometry. Among them, we enlist PDE problems, including geometric flows, metric geometry and geometric variational problems.

The main objective of the proposed Workshop was to gather many leading experts in these various fields and younger brilliant researchers in order to foster the exchange of in the circle of techniques and ideas previously described.

Moreover, many of the speakers work in Mathematical General Relativity; in fact, on the one hand, this topic already proved and still promises to be a source of very stimulating and important mathematical problems, and, on the other hand, it enjoys contributions from the other fields.

The meeting has been undoubtedly successful. In fact, the talks, themselves of high quality, fostered very rich moments of discussion. The have taken place both after the very end of the talks and in the various moments of the day that were devoted to discussions.

The meeting, most importantly, has been a fruitful moment of scientific exchange between the Chinese community in Geometric Analysis and the Western community in Geometric Analysis.

The participants have discussed with particular emphasis the open problems and the issues that have been suggested in the talks, as well as other topics connected to these. Examples include the functional and geometric properties of PDEs with very strong geometric content, including geometric flows, that arose e.g. from the talks of Mari, Pluda, Espinar, Benatti, Xia, Munteanu and Mazzieri,the comparison geometry related to scalar curvature, related e.g. to the talk of Shi and Hirsch, the comparison geometry for new curvature invariants, connected with the talk of Johne, Wu, Wang, the shape of possibly nonsmooth manifolds with a notion of curvature lower bounds, a topic connected to the talks e.g. of Antonelli, Xu, Gigli, Sormani, Lee, Zhu, Graf and Burkhardt-Guim and problems in the comparison geometry of curvature bounds arising from General Relativity, as discussed e.g. by Borghini, Wiygul, Khuri and Reiris.

We finally mention that the paper [2] arose from discussions the authors had during this workshop, and that the paper [1] was completed benefiting from discussions the authors had with Luca Benatti, Luciano Mari and Kai Xu during the workshop.

References

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