Determining a Collection of Boundary Points on the p-adic Mandelbrot Set of degree d Polynomials

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1 Introduction and Background

In arithmetic dynamics we are interested in understanding the behavior of points under iteration of a rational map f. That is, we are interested in the set $\mathcal{O}_f(\alpha) = \{f^n(\alpha) : n \in \mathbb{Z}_{\geq 0}\}$ where f^n is the n^{th} iterate of f.

We say a map is post-critically bounded (PCB) if the set $\mathcal{O}_f(\alpha)$ is bounded for all critical points α of f, and post-critically finite (PCF) if $\mathcal{O}_f(\alpha)$ is finite for all critical points α of f. The famous Mandelbrot set is defined as

$$\{c \in \mathbb{C} : f_c(z) = z^2 + c \text{ is PCB}\}.$$

Parameters c for which 0 is strictly preperiodic for $z^2 + c$ are called Misiurewicz points, all of which appear on the boundary of the Mandelbrot set. The Mandelbrot set is self-similar as one zooms in on a Misiurewicz point, giving it its famous fractal-like structure. Moreover, the Mandelbrot set near a Misiurewicz point is similar to the Julia set for the associated polynomial, revealing key information about the dynamical behavior of that polynomial.

In the p-adic setting, the analogue of the Mandelbrot set $\{c \in \mathbb{C}_p : z^2 + c \text{ is PCB}\}$ is simply the unit disk; however the set is more interesting when we generalize this notion to higher-degree polynomials. We define the p-adic, degree d Mandelbrot set as follows:

$$\mathcal{M}_{d,p} := \{(a_{d-1}, \ldots, a_1, a_0) \in \mathbb{C}_p : z^d + a_{d-1}z + \ldots a_1 z + a_0 \text{ is PCB}\}.$$

When $p \geq d$, this generalized Mandelbrot set is simply the unit polydisk, but when $p < d$ its structure is much more complicated and reminiscent of the complex Mandelbrot set. In general, very little is known about the structure of the p-adic degree d Mandelbrot set when $p < d$. In [**?**], Anderson showed that the polynomial

$$f_t(z) = z^3 - \tfrac{3}{2}tz^2$$

for $t = 1$ is a point on the boundary of the 2-adic degree 3 Mandelbrot set. Moreover, when zooming in on this parameter, one sees self-similar behavior, much like in the complex setting. Then, in [**?**] we established a point on the boundary of the 2-adic degree 3 Mandelbrot set associated with the map

$$f_t(z) = -\frac{3}{2}t(-2z^3 + 3z^2) + 1 \tag{1}$$

when $t = 1$. Both are examples of one-parameter families of cubic polynomials with just one critical orbit to study: in the first case, one of the two critical points is fixed, and in the latter case, one of the two critical points maps to the other.

There are two natural extensions of this result which we hope to prove. First, we want to generalize the one-parameter family of polynomials described in Equation 1 to higher degrees. Let $d \geq 3$ and $p \mid (d-1)$. Consider the family of polynomials defined by

$$\mathcal{F}_{d,t}(z) = \tfrac{d}{d-1} t \left((d-1)z^d - dz^{d-1} + 1 \right). \in \mathbb{C}_p[z]. \tag{2}$$

Notice that $\mathcal{F}_{d,t}(z)$ has critical points 0 and 1. Furthermore,

$$\mathcal{F}_{d,1}(z) = -\tfrac{d}{d-1} \left(-(d-1)z^d + dz^{d-1} \right) + \tfrac{d}{d-1}$$

is post-critically finite with $f_1(1) = 0$ and $f_1(0) = \frac{d}{d-1}$, which is repelling a fixed point. This is a generalization of the cubic family from Equation 1, which has similar critical behavior.

Question 1. *For any degree $d \geq 3$ and $p \mid (d-1)$, is $\mathcal{F}_{d,1}$ a non-isolated boundary point on the p-adic Mandelbrot set?*

Answering Question 1 would establish a point on the p-adic degree d Mandelbrot set for all degrees d and primes $p \mid (d-1)$, which would be signifiant progress in the area.

Additionally, we would like to explore the set of PCF cubic polynomials in the parameter space of all cubic (non-unicritical) polynomials.

Question 2. *Every cubic polynomial is conjugate to one of the form $\mathcal{G}_{r,s}(z) = z^3 - \frac{3}{2}(r+s)z^2 + 3rsz$, whose critical points are r and s. Suppose (r_0, s_0) is such that the corresponding polynomial $\mathcal{G}_{r_0,s_0}$ is PCF, and $\mathcal{G}^n_{r_0,s_0}(r_0) = 0$ for some n, where 0 is a repelling fixed point. In this case, $\mathcal{G}_{r_0,s_0}$ is analogous to a Misiurewicz point on the classical Mandelbrot set. Do these polynomials all lie on the boundary of $\mathcal{M}_{3,2}$?*

Throughout this note, we will be working over $\mathbb{C}_p$ and we will write $|\cdot| = |\cdot|_p$ is the normalized p-adic absolute value.

2 Progress Made at BIRS

During our time at BIRS we focused on Question 1. We were able to establish a complete result in the case where $d = p + 1$ for some prime p, and make progress towards a result in the more general case where $p \mid (d-1)$. In order to prove that $t = 1$ corresponds to a point on the boundary of the p-adic Mandelbrot set, we must show that there are both parameters t arbitrarily close to 1 (in the p-adic topology) yielding post-critically bounded polynomials and yielding post-critically unbounded polynomials.

We established that for $p \mid d-1$ there are parameters t arbitrarily close to 1 that yield post-critically unbounded polynomials, $\mathcal{F}_{d,t}(z)$.

Theorem 1. *Consider the following one-parameter family of degree $d \geq 2$ polynomials in $\mathbb{C}_p[z]$:*

$$\mathcal{F}_{d,t}(z) = \tfrac{d}{d-1} t \left((d-1)z^d - dz^{d-1} + 1 \right).$$

Suppose $v_p(d-1) = e$, where e is a positive integer. Suppose further that $|t-1| = p^{-k}$, where $k = qe(d-1) + r$ such that q, r are positive integers and $r < e(d-1)$. Then, $F_{d,t}(z)$ is post-critically unbounded.

The methods used to prove this result were generalized from our earlier result in [**?**]. In the case where $d = p + 1$, we were able to obtain the complete result that $\mathcal{F}_{d,1}$ corresponds to a point on on the p-adic Mandelbrot set, by establishing values of t arbitrarily close to 1 corresponding to polynomials $F_{d,t}$ that are post-critically bounded. In general, it is significantly more challenging to establish boundedness. We were able to obtain this result using a Newton polygon argument that establishes values t_n approaching 1 for which the critical orbit of $\mathcal{F}_{d,t_n}$ is of length n.

Theorem 2. *There exists a sequence $\{t_n\}_{n=3}^{\infty}$ such that $\lim_{n\to\infty} t_n = 1$ and $\mathcal{F}_{d,t_n}$ is PCF with a periodic critical orbit of length n. Moreover, for $t \in \overline{D}(t_n, p^{-r})$ for $r = (n-2)p + k + 1$, the polynomial $\mathcal{F}_{d,t}(z)$ is PCB.*

While Theorem 2 establishes the desired result, it does not establish information regarding a pattern of disks corresponding to PCB polynomials. We worked to determine specific patterns for values of t that correspond to PCB polynomials by finding cycles of disks. We have obtained the following partial results.

Lemma 1. *Let $t \in D\left(1 + (p-1)p^p, p^{-(p+3)}\right)$. If $z \in D(0, p^{-2})$ or $z \in D((p-1)dp^{p-1} + \frac{d}{p}, p^{-(p+2)})$ then $\mathcal{O}_{\mathcal{F}_{d,t}}(z)$ is bounded, and therefore $\mathcal{F}_{d,t}$ is PCB. In particular, these disks are in a 2-cycle.*

Lemma 2. *If $|t| = 1$, then $F_{d,t}$ maps the disk $D(1, p^{-k})$ to $D(0, p^{-2k})$ for any positive integer k.*

In addition to these partial results, we generated data that we believe will lead to a generalization of Lemma 1 for cycles of length n, which will hopefully establish a self-similar disk pattern corresponding to the periodic orbits found in Theorem 2.

3 Future Work

There are three research areas we intend to pursue in the future. First, we would like to continue our work towards understanding the structure of the p-adic degree d Mandelbrot set in the case where $d = p + 1$. In particular, we would like to determine patterns for the parameters t that yield PCB polynomials. While Theorem 2 establishes the existence of parameters t arbitrarily close to 1 with a critical orbit of length n for all n, understanding where these values are can provide an understanding of how the p-adic Mandelbrot sets are analogous to the complex Mandelbrot sets.

Beyond that, we plan to further pursue the more general case of when $p \mid (d-1)$ with hopes to establish that $\mathcal{F}_{d,1}(z)$ corresponds to a point on the boundary of the p-adic degree d Mandelbrot set. As we already have Theorem 1 establishing post-critical unboundedness in maps $\mathcal{F}_{d,t}$ for values of t arbitrarily close to 1, we are now tasked with establishing values of t arbitrarily close to 1 for which $\mathcal{F}_{d,t}$ is post-critically bounded. While the Newton polygon argument used in Theorem 2 doesn't seem to carry over to this case, our hope is that understanding a pattern of values of t with particular post-critical behavior will provide insight to the more general $d = p + 1$ case.

Additionally, while we spent our time at BIRS working towards Question 1, we would still like to explore Question 2.