Research in teams

The Steklov eigenproblem under polygonal and polyhedral approximation

1 Overview

Our goal for this RIT was to bring together expertise in numerical analysis and spectral geometry, to study the impact of domain approximations on the Steklov spectrum.

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \in \{2,3\}$. Let $H^1(\Omega)$ be the Sobolev space $W^{1,2}(\Omega)$. Endowed with the inner product $(u,v)_{H^1(\Omega)} = (u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}$, where $(\cdot,\cdot)_{L^2(\Omega)}$ is the usual L^2-inner product $(u,v)_{L^2(\Omega)} = \displaystyle\int_\Omega u\,v$, the space $H^1(\Omega)$ is a Hilbert space. The Steklov problem can be stated in weak form as: *find $u \neq 0 \in H^1(\Omega)$, $\lambda \in \mathbb{R}$ such that*

$$\int_\Omega \nabla u \cdot \nabla v = \lambda \int_{\partial\Omega} uv, \qquad \forall v \in H^1(\Omega). \tag{1}$$

Under fairly mild conditions on the boundary regularity of Ω (eg. Lipschitz), it is well known that the Steklov spectrum is countable, accumulates only at infinity, and consists of non-negative real values.

The eigenvalue problem cannot be solved in closed form except for some special domains. In most instances, a provably convergent numerical approximation of solutions to (1) must be devised and computed.

The finite element approach is a discretization scheme that relies on the variational characterization of eigenvalue problems, and allows for provable error and approximation estimates. In this approach, a domain is tesselated by (non-overlapping, usually simplicial) subdomains, and polynomial spaces on the subdomains with some global properties are used to achieve approximation of eigenfunctions. Denote by Ω_h the union of these subdomains.

Recent work in numerical analysis has focused on finite element approximation of the Steklov spectrum for polygonal and polyhedral domains since they are naturally tesselated by simplices. In this case, there is no *variational crime* since the domain of interest is precisely the union of the subdomains. When this is not the case, obtaining precise rate of convergence is harder, especially uniformly in the eigenvalues and eigenfunctions. The characterisation of the convergence properties of finite element methods for smooth domains is of crucial importance for the Steklov eigenproblem – most of the interesting behaviour of the eigenfunctions happens close to the boundary.

Since both the analytico-geometric and finite element approaches are framed within the variational paradigm, we aim at combining these two approaches to reduce the drawbacks from either. More broadly, there are technical questions in each of the fields of spectral geometry and numerical analysis which could be answered using techniques from the other.

2 Objectives for the workshop

The goal of this RIT activity was to study an open question posed in an AIM workshop in 2018 which links numerical analysis and spectral geometry:

Question 1: If we use a finite element approximation via polygonal domains Ω_h to a smooth domain Ω , in what precise sense do the Steklov eigenvalues of Ω_h converge to those of Ω_h?

Fundamental recent results in spectral geometry tell us the very different rates at which the spectra of smooth domains and those of polygonal domains approach certain *quasimodes*. The implication in numerical analysis is to raise the following question: if we use a finite element approximation via polygonal domains to a smooth domain, in what precise sense do the spectra of the approximants converge to the actual one? Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $\Omega_h \subset \mathbb{R}^d$ be a polygonal approximating domain in the sense that $\Omega_h \to \Omega$, as a mesh parameter $h \to 0$. Let $\{\sigma_k(\Omega)\}_{k=1}^{\infty}, \{\sigma_{k,h}(\Omega_h)\}_{k=1}^{N_h}$ be the Steklov spectra on Ω, Ω_h respectively. In the situation where Ω itself is polygonal, we can consider $\Omega_h = \Omega$. Then, finite element approximation via triangular elements (affine maps of a reference element $\hat{K}$), using conforming elements based on the finite element triples based on p^{th} degree polynomials $(\hat{K}, P_p(\hat{K}), \Sigma_p(\hat{K}))$ are well characterized and one can show, for instance, that under reasonable assumptions on the boundary,

$$|\sigma_k(\Omega) - \sigma_{k,h}(\Omega_h)| = |\sigma_k(\Omega) - \sigma_{k,h}(\Omega)| \leq C_k(\Omega)h^{2p}.$$

Here $C_k(\Omega)$ may depend on the number k and the regularity of the eigenfunction associated with the k^{th} eigenvalue $\sigma_k(\Omega)$.

In the interesting situation where $\Omega \neq \Omega_h$, the rate above is not guaranteed, since

$$\int_{\Omega} u \, d\Omega \neq \int_{\Omega_h} u \, d\Omega,$$

for functions u defined on the union $\Omega \cup \Omega_h$ of the two domains. What is the precise theoretical characterization of the *variational crime* during the eigenvalue computation? In fact, given the assumption that $\Omega_h \neq \Omega$, we notice the following

$$|\sigma_k(\Omega) - \sigma_{k,h}(\Omega_h)| \leq |\sigma_k(\Omega) - \sigma_k(\Omega_h)| + |\sigma_k(\Omega_h) - \sigma_{k,h}(\Omega_h)|.$$

Given that, as discussed above, error estimates for Steklov eigenvalues are well understood for polygonal domains, we see that the second term on the right hand side of the inequality above can be bounded by

$$|\sigma_k(\Omega_h) - \sigma_{k,h}(\Omega_h)| \leq C_k(\Omega_h)h^{2p}.$$

Now, the question that arises is, what error estimates can be obtained for the term $|\sigma_k(\Omega) - \sigma_k(\Omega_h)|$? open question posed in an AIM workshop in 2018, and remains an open important question.

The end-goal of this team activity is to solve this problem. Our team involves two spectral geometers and two numerical analysts, and our initial objective is to inform each other of the precise state-of-the-art in our respective fields as relevant to this question. We then aim to use Strang-type arguments informed by results from spectral geometry to state and prove the critical theorem.

3 RIT outcomes

This RIT was the first such collaborative activity between the participants from numerical analysis and spectral geometry on the Steklov problem using polygonal domains.

A major outcome of this RIT was to develop an initial shared vocabulary and to identify a collection of open problems to study. Some of questions are specific to the Steklov problem. Other questions emerged as we realized that techniques from geometry and analysis could address particular questions in numerical analysis.

List of questions identified. In what follows, let $\Omega \subseteq \mathbb{R}^d, d = 2, 3$ be a simply connected, contractible bounded domain with smooth boundary. Let $h > 0$ be the mesh parameter, and the polyhedral approximating domain be Ω_h.

1. **Variational crime:** Let $\Omega \subseteq \mathbb{R}^2$, and fix eigenvalue index k. Let $\{\Omega_n : n \geq 3\}$ be a polygonal exhaustion of Ω by n-gons. If Ω is a disk and Ω_n is a regular n-gon , the Steklov eigenvalues satisfy for all $k \geq 0$
$$|\sigma_k(\Omega_n) - \sigma_k(\Omega)| \lesssim_k \frac{n}{\log n}. \tag{2}$$
What is the rate in the following situations?
   - Ω a disk, Ω_n an $n - gon$ (not necessarily regular)
   - Ω a smooth domain
2. **Finite element approximation error:** Fix an eigenvalue index range $k \in [0, k_{max}]$. Using finite element approximating polynomials of degree p, find the rate of convergence r of for the finite element approximation $\sigma_{k,h}(\Omega_h)$ to $\sigma_k(\Omega)$. More precisely, can one find rate r and constant $C(k, p, \Omega)$ so that for each $k \in [0, k_{max}]$,
$$|\sigma_k(\Omega) - \sigma_{k,h}(\Omega_h)| \leq C(k, p, \Omega)h^r?$$
3. **Isogeometric/Spline approximants of boundary:** The two previous questions are linked. Generally, finite element methods place points on the boundary of Ω and connect these with straight lines, that is a linear interpolation is used between any two points on the boundary.

   Suppose $\{\Omega_n : n \geq 3\}$ is an exhaustion of Ω using high-order spline approximations of $\partial\Omega$. Fix eigenvalue index k. As $n \to \infty$, what is the rate $f(n)$ and the constant $C(p, k)$ in the estimate
$$|\sigma_k(\Omega_n) - \sigma_k(\Omega)| \lesssim_k C(p, k)f(n) \tag{3}$$
4. **Lagrange interpolation constant:** An important quantity in approximation theory is the Lagrange interpolation constant. Let K be a triangle with vertices v_1, v_2, v_3. We denote by e_j the edge opposite to v_j and by α_j the unit vector parallel to e_j.

   Let Π_1 be the Lagrange interpolant, i.e. the map $\Pi_1 : \mathrm{W}^{2,2}(K) \to P_1$ that sends f to the linear function $\Pi_1(f)$ whose values at the vertices agree with those of f. We want to estimate the Lagrangian interpolation constant
$$C_1(K) = \sup\left\{\frac{\|\nabla f\|_{L^2(K)}}{\|\nabla^2 f\|_{L^2(K)}} : f \in \ker(\Pi_1)\right\} \tag{4}$$
   There are numerous upper bounds in the literature, which are valid under conditions on the triangles. Can we use techniques from spectral geometry to get either universal upper bounds or easily-computable bounds on curvilinear triangles?
5. **Higher-order interpolation:** For interpolation by higher-order polynomials on a triangle K, we wish to estimate the interpolation constants for $m \leq \ell - 1$
$$C_m(K) = \sup\left\{\frac{\|f\|_{H^{m-1}(\Omega)}}{|f|_{H^m(\Omega)}} : f \in \ker(\Pi_\ell)\right\} \tag{5}$$
   where $\Pi_\ell : \mathrm{W}^{2,2}(K) \to P_\ell$. Can this be bounded in terms of explicitly computable isogeometric quantities? The question is inspired by a result on interpolation onto constants. In this case, the interpolation constant is bounded in terms of the first non-zero Neumann eigenvalue on the triangle, which in turn can be bounded in terms of $j_1, 1$.

We have made progress already on questions 1 and 4.