Combinatorics and Geometry of Moduli Spaces of Curves

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1 Overview of the Field

Summary. Algebraic curves are among the most ubiquitous mathematical objects; they have deep connections to algebra, geometry, topology, and mathematical physics. Moduli spaces of curves allow one to study curves in families, whereby each point of the moduli space corresponds to a curve. These spaces exhibit rich combinatorial and recursive structure, which often allows problems about curves and their moduli spaces to be approached using tools from classical enumerative and modern algebraic combinatorics, including generating functions, Young tableaux, polytopes, and graph theory.

Recent years have seen rapid advances on geometric questions about curves and their moduli spaces, many of which have introduced new and intriguing combinatorial objects of study. For instance, one celebrated result on curves is the fact that there is no way to parametrize a family of general smooth curves of genus $g \geq 22$ in terms of free parameters. The proof of this fact combined the study of the moduli space M_g of smooth curves of genus g, together with combinatorial methods: for $g \geq 23$, examining extremal rays of cones of effective divisors [15, 30]; and for $g = 22$, using Young tableaux, graphs, and new ideas in tropical geometry [17].

The role of combinatorics in moduli of curves. A fundamental feature of moduli spaces of curves, which is also a source of combinatorial structure, is self-similarity. By marking points on the curves, we get the moduli spaces $\overline{M}_{g,n}$ of stable n-pointed curves of genus g. These spaces are interconnected for different values of g and n through tautological forgetful and gluing maps, and the boundary of $\overline{M}_{g,n}$, which parametrizes singular curves, is self-similar: it is stratified by products $\prod \overline{M}_{g',n'}$ for smaller g' and n' (modulo a finite symmetric group action). Each stratum parametrizes curves with a specified topological type, encoded by a *dual graph* indicating the components, genera, marked points, and nodes. Graphical properties are also used to describe a range of closely related moduli spaces [1, 3, 31, 49, 50, 51].

Many core questions about $\overline{M}_{g,n}$ and its cousins reflect these combinatorial and graphical underpinnings. For example, the *tautological ring* $R_{g,n}$, introduced by Mumford [42], has been partly described by Pixton [47] using generators and relations that are formal sums of decorated dual graphs. These rings are generated by classes obtained by pushing and pulling basic cohomology classes along the tautological maps, and while $R_{g,n}$ is much smaller than the Chow ring, it has a rich structure reflecting deep geometric information. Generators for $R_{g,n}$ have been explicitly described; relations, proposed by Faber and Faber–Zagier, have been shown in special cases [10, 43, 44], and generalized further [48]. The full system of relations is still unknown and has since become one of the most important open problems in the field. Indeed, since the tautological rings were first defined in the 1980s, there have been more than 130 published papers on the subject, with more than 60 having appeared in the last 10 years.

A second important example is the use of tropical methods: tropical geometry works directly with metric graphs and is an ongoing and active (parallel) research focus. In addition to the breakthrough result above in genus 22, tropical methods have been used to calculate certain cohomology groups of $\overline{M}_g$ using graph homology [8], and tropical methods famously led to a proof of the (classical) Brill–Noether Theorem [12], which examines the geometry of curves via their projective embeddings. Indeed, the past decade has seen a resurgence of activity in Brill–Noether theory: in addition to the tropical proof of the classical Brill–Noether theorem, recent work has examined curves with low gonality [33, 46], Brill–Noether theory and Hurwitz spaces [40], and the breakthroughs of Farkas–Jensen–Payne on the Kodaira dimension of $\overline{M}_g$ and of Larson–Vogt [41] on the interpolation problem for Brill–Noether curves. These results all rely on nontrivial combinatorial and especially tropical arguments, which suggest the potential for further advances with more sophisticated combinatorics.

A third example comes from the facts that questions about curves of genus $g > 0$ can often be reduced to questions in genus 0, and that $\overline{M}_{0,n}$ shares some (but not all) features with simpler spaces such as toric varieties. As such, $\overline{M}_{0,n}$ itself is an important test variety in moduli theory, birational geometry, and other areas of algebraic geometry. A key question about $\overline{M}_{0,n}$, for example, is to determine for which n it is a *Mori Dream Space* (MDS) – a variety whose birational geometry is essentially the nicest possible. All toric varieties are Mori Dream Spaces, whereas $\overline{M}_{0,n}$ is known to be an MDS for $n \leq 6$, and was recently shown *not* to be an MDS for $n \geq 10$ [32]. Polyhedral and other combinatorial methods played an important role in these and related results. These results are closely related to the F-Conjecture [24, 35] that the nef cone of $\overline{M}_{g,n}$ is a polyhedral cone. This conjecture holds for $\overline{M}_{g,n}$ if it holds for S_g-invariant divisors on $\overline{M}_{0,g+n}$. Using this theorem the F-Conjecture is known on $\overline{M}_g$ for $g \leq 35$ [18, 23], where the most recent jump from $g \leq 24$ to $g \leq 35$ was achieved by Fedorchuk with a completely combinatorial argument.

Other recent work on moduli of curves has made use of combinatorial tools including k-cores of Young tableaux and the affine symmetric group [40], simplicial complexes [11, 19], parking functions [6] and new enumerations over trees [25].

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2 Structure of the workshop

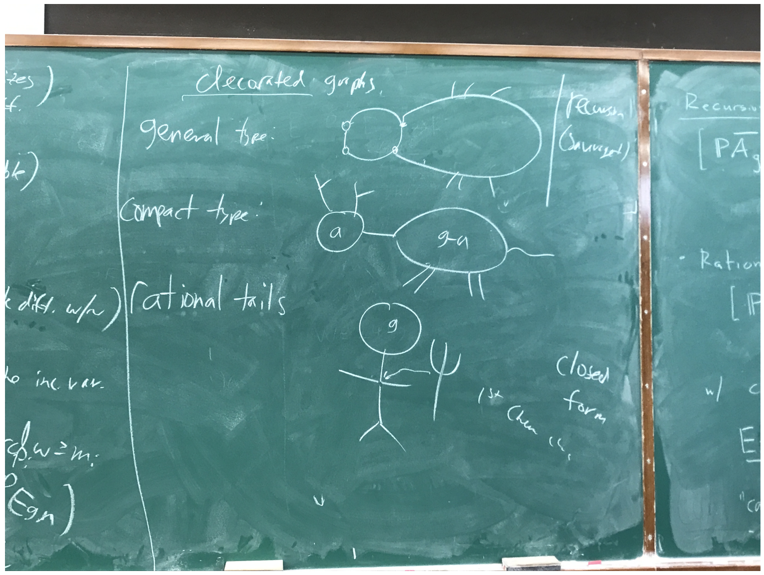
The purpose of our workshop, entitled *Combinatorics of Moduli of Curves* (COMOC), was to bring together two groups of expert researchers: geometers studying the moduli space of curves who work on combinatorial questions, and combinatorialists interested in developing solutions to these problems. Participants were split into working groups of 4-5 people with a mix of mathematical specialties, led by one or two established researchers with expertise in the moduli space of curves who proposed a problem to work on together. These teams collaborated in their working group sessions throughout the week. Our main goals were

- to introduce the combinatorialists to new sources of problems, and to expose the geometers to new relevant combinatorial techniques,
- to spark new collaborations and professional connections across these two groups.

Each of the core aspects of our workshop – the research focus, invited working group leaders and participants, talks, and other activities – was directed towards these two objectives. Notably:

- To establish a common language among *all* participants, we provided a resource page on our conference website with background notes and references on basics about the moduli space of curves. We also surveyed the participants on their background knowledge pertaining to moduli spaces of curves, and shared the results with the working group leaders.
- To foster research *within* each group, we asked each leader to write a project description for a specific combinatorial problem stemming from their work. We distributed these project descriptions to the participants and formed them into working groups. When assigning groups, we balanced considerations of career stage, research background, and participants' own preferences collected via a second survey.
- To encourage the transfer of ideas in *both* directions between combinatorics and geometry, and to set the stage for our workshop, we opened the week with introductory talks by two of the co-organizers. The first talk, by Levinson, gave an overview of the 'zoo' of objects and constructions that the group projects would be focusing on during the week: e.g. marked curves, stable graphs, stable maps. It set up the geometry–combinatorics interplay between these objects, themes that were echoed in later presentations during the week. The second talk, by Gillespie, introduced the combinatorial technique of *sign-reversing involutions* and illustrated how to apply it solve a combinatorial problem inspired by moduli spaces of marked curves. In a spirited Q&A session after the talk, participants started discussing the geometric meaning of the calculation, and soon realized that Gillespie and Levinson's sign-reversing involution gave a short proof of a formula for the Euler characteristic of the link of $M_{0,n}^{\mathrm{trop}}$, the tropical moduli space of genus zero curves. The talks and ensuing discussions had the intended affect of sparking a dialogue that crossed the boundary of combinatorics and algebraic geometry, and that continued throughout the week.
- To further discussion *across* groups, leaders gave short 5-7 minute descriptions of their projects to the whole workshop on the morning of the second day. We also held professional development activities and social activities throughout the week, which helped bring participants together into a cohesive group: a game night, discussions on communication across geometry and combinatorics and on mentorship, and a group hike.

To cap off the workshop, group representatives gave short presentations on the Friday morning about progress made during the week and plans for continuation of the work. Monday's 'zoo' of objects, spaces and constructions reappeared in more than one closing presentation:



Many groups also expressed an intention to continue working on the projects. To our knowledge, at least four of the eight groups have actively pursued the projects begun at COMOC.

Participant feedback. We solicited anonymous feedback from participants following the conclusion of the conference. We received many positive comments, including:

- *I particularly enjoyed the positive and inclusive environment supported by this community.*
- *In my opinion, it was the right decision to devote almost all of the time to group work, and to allow the individual groups to structure their time as they saw fit.*
- *I've never participated in a workshop like this, but I am now a super fan. I thought the structure and format of the conference were fantastic, giving groups a great deal of time to work together and make progress on their projects. I also think it was really nice to have a common schedule, allowing participants from different groups to mingle during coffee breaks and meals.*
- (In response to "What future activities would you like to see to further the connections between the communities we brought together at COMOC?") *More of the same!*

3 Project summaries

3.1 PL functions and Minkowski weight on $M_{0,n}^{\mathtt{trop}}$

Project leader: Renzo Cavalieri

There are two natural ways to describe a line bundle on $\overline{M}_{0,n}$:

1. as associated to a linear combination of boundary divisors;
2. by specifying the degree of its restriction to each boundary curve.

The two descriptions are related (and equivalent) by Poincaré duality. But it is not so straightforward to go from one to the other (at least systematically, or for infinite families of line bundles one for each n) because boundary divisors and boundary curves do not form bases of their respective Chow groups: they generate, but there are a ton of WDVV relations. Both description admit a tropical version. Perspective 1 is given by piecewise linear functions - where the coefficients of the linear combination become the slopes along the corresponding ray; Perspective 2 by Minkowski weights, where codimension one cones are given weight equal to the degree of the restriction to the corresponding boundary stratum. In this project participants explored how, given a set of balanced Minkowski weights, can one explicitly exhibit a PL function corresponding to the same line bundle.

Over the course of the week, the group was able to find a combinatorially described basis for $\mathrm{Pic}(\overline{M}_{0,n})$, together with a dual basis, where the dual to a particular irreducible boundary divisor is given as a linear combination of F-curves: a simple combinatorial rule gives the coefficients of this linear combination.

Given these results, the group turned their attention to possible applications; in particular we focused on the following two:

Hodge classes on admissible covers. We aim to compute the pushforward to $\overline{M}_{0,n}$ of λ classes, Hurwitz-Hodge classes and other tautological classes on moduli spaces of admissible covers. A lot has been done for λ_1 and for moduli spaces of cyclic covers, but for either the cases of higher degree lambdas or admissible covers with a non-abelian structure group, while computing the strata intersections is reasonably straightforward and the intersections have a lot of symmetry, the complexity of inverting the intersection matrix is essentially what prevents to give satisfactory answers (i.e. to give systematic formulas for infinite families of spaces).

Tropical intersection theory. We aim to use tropical intersection theory to compute the intersection numbers of the tropicalization of effective divisors that are not dimensionally transverse to the boundary with all F-curves.

3.2 Moduli Theory of the r-Braid Arrangement

Project leaders: Emily Clader and Dusty Ross

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{CP}^n$, and let $\mathcal{L}_\mathcal{A}$ denote the intersection lattice of $\mathcal{A}$—that is, the set of all intersections of subsets of $\mathcal{A}$. A **wonderful compactification** of $\mathcal{A}$ is, roughly speaking, a way of replacing $\mathbb{CP}^n$ by a different ambient variety in such a way that the complement of the hyperplanes is preserved but the arrangement $\mathcal{A}$ itself is replaced by a divisor with normal crossings. There are a number of such compactifications, given by blowing up $\mathbb{CP}^n$ (in a carefully-prescribed order) along subsets $\mathcal{G} \subseteq \mathcal{L}_\mathcal{A}$ known as **building sets**. The collection of building sets is partially ordered by inclusion, and it has a unique maximal and minimal element.

In the case where $\mathcal{A}$ is the **braid arrangement** consisting of the hyperplanes

$$H_{ij} := \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{CP}^n \mid x_i = x_j\}$$

for all $0 \leq i < j \leq n$, and $\mathcal{G}$ is the minimal building set, the resulting wonderful compactification is the moduli space of curves $\overline{\mathcal{M}}_{0,n+3}$. On the other hand, for any positive integer r there is a natural generalization of the braid arrangement, which we refer to as the **r-braid arrangement**, consisting of the hyperplanes

$$H_{ij}^k := \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{CP}^n \mid x_i = \zeta^k x_j\}$$

for all $0 \leq i \leq j \leq n$ and all $0 \leq k \leq r-1$, where ζ denotes a primitive rth root of unity.

Problem: Describe the wonderful compactification of the r-braid arrangement, with its minimal building set, as a moduli space of curves.

1. We studied the combinatorics of the intersection lattice of the 2-braid arrangement in order to show that the desired wonderful compactification can be concretely described as the blow-up of $\mathbb{CP}^n$ along all proper subspaces of the form

$$\{[x_0 : \cdots : x_n] \in \mathbb{CP}^n \mid x_{i_1} = \pm x_{i_2} = \cdots = \pm x_{i_\ell}\},$$

in increasing order of dimension, and we conjectured a similar description for all r.

2. We gave a candidate description of the moduli problem for which we believe this wonderful compactification to be a fine moduli space, which is closely related to the space constructed in [CDH+22, CDLR23] parameterizing rational curves with an order-r automorphism under which the marked points form a collection of orbits.

We also laid out a proposed strategy for proving that the blow-up in item (1) above is indeed a fine moduli space for the moduli problem in item (2), by re-interpreting the blow-up as a fiber product and using universal properties to construct the universal family. We plan to continue meeting virtually in the coming months to carry out this strategy.

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3.3 Kleiman and the F-Conjecture

Project Leader: Angela Gibney

Our project concerns an old question about divisors and curves on $\overline{\mathrm{M}}_{g,n}$, the moduli space of stable n-pointed curves of genus g [KM96], [GKM02]. The F-conjecture asserts that every curve on $\overline{\mathrm{M}}_{g,n}$ is numerically equivalent to a finite effective sum of so-called F-curves. These curves are defined as (numerical equivalence classes) of the images of certain clutching maps from the one dimensional spaces $\overline{\mathrm{M}}_{0,4}$ and $\overline{\mathrm{M}}_{1,1}$. Another way to state the conjecture is to say that there are finitely many extremal rays of the Mori cone of curves, each of which is spanned by an F-curve.

One application, given that there would be finitely many extremal rays of the closed cone of curves if the conjecture holds, would be the potential to classify all morphisms from $\overline{\mathrm{M}}_{g,n}$ to other projective varieties. Some researchers believe strongly that the F-conjecture holds on $\overline{\mathrm{M}}_{g}$, for all g, as it is known to hold for $g \leq 35$ [GKM02, Gib09, Fed20]. On the other hand, the conjecture is not supported by the main structure theorems from Mori theory, and there is some doubt about whether the assertion is true more generally as it has been verified on $\overline{\mathrm{M}}_{0,n}$ only for $n \leq 7$ [KM96].

By [24], the conjecture on $\overline{\mathrm{M}}_{0,n+1}$ is equivalent to the conjecture on $\overline{\mathrm{M}}_{1,n}$, and recently, in new work from [Gib24], divisors that contract certain collections of F-curves on $\overline{\mathrm{M}}_{1,n}$ are shown to be nef. In particular, there is a new reduction of the conjecture to checking nefness of divisors on $\overline{\mathrm{M}}_{1,n}$ that contract certain F-curves.

F-curves, F-divisors, and the tools used to work with them can be given in purely combinatorial terms. Working in this framework, and using Kleiman's nefness criteria, our group is looking for support for the conjecture, or the potential that it fails on $\overline{\mathrm{M}}_{1,7}$, which is the first unknown case.

To describe the conjecture and our project in more detail, a few definitions will be given.

A divisor on $\overline{\mathrm{M}}_{g,n}$ that non-negatively intersects all F-curves is called an F-divisor. A divisor is nef if it non-negatively intersects all curves. F-divisors and nef divisors form cones in $\mathbb{R}^{\rho}$, where ρ is the Picard number of $\overline{\mathrm{M}}_{g,n}$. The F-Conjecture on $\overline{\mathrm{M}}_{g,n}$ predicts that a divisor on $\overline{\mathrm{M}}_{g,n}$ is nef if and only if it is an F-divisor. In other words that the F-Cone is equal to the cone of nef divisors.

Let $\mathrm{F}_{g,n} : \overline{\mathrm{M}}_{0,g+n} \to \overline{\mathrm{M}}_{g,n}$ be the map which associates to a $g+n$ pointed rational curve, the n pointed curve of genus g obtained by gluing g fixed curves, corresponding to points $(E, q) \in \overline{\mathrm{M}}_{1,1}$, to it, via the identification of the point q to each marked point. By [24, Theorem 0.3], an F-divisor D on $\overline{\mathrm{M}}_{g,n}$ is nef if and only if $\mathrm{F}^*_{g,n}(D)$ is nef, and the F-Conjecture on $\overline{\mathrm{M}}_{g,n}$ is equivalent to an S_g-invariant version of the F-Conjecture on $\overline{\mathrm{M}}_{0,g+n}$. So for instance, the F-Conjecture on $\overline{\mathrm{M}}_{0,n+1}$ is equivalent to the F-Conjecture on $\overline{\mathrm{M}}_{1,n}$, and the S_{n+1}-invariant F-Conjecture on $\overline{\mathrm{M}}_{0,n+1}$ is equivalent to the the F-Conjecture on $\overline{\mathrm{M}}_{n+1}$.

Kleiman's nefness criteria [Laz04, Theorem 1.4.9], stated in terms of $\mathbb{R}$-divisors, says that a divisor D on a variety X is nef if and only if $\mathrm{D}^{\dim(\mathrm{V})} \cdot \mathrm{V} \geq 0$ for every closed subvariety $V \subset X$. To apply Kleiman's criteria to investigate the F-Conjecture, one can ask if D is an F-divisor, whether the top self-intersection of D is nonnegative. If so, this lends support for the conjecture, and if not, this shows the conjecture is false. In this project we are considering this question on $\overline{\mathrm{M}}_{1,7}$ for divisors D that lie on extremal faces of the cone of F-divisors that are not known to be nef via the criteria given in [Gib24].

The goal is to translate the problem to determining the positivity or negativity of an explicit homogeneous polynomial corresponding to the top self intersection of a given F-divisor. Our approach has two parts: First, to produce the polynomial, and second, to analyze its positivity using numerical methods.

So far we have produced the homogeneous polynomials for $\overline{\mathrm{M}}_{1,n}$ for $n \leq 7$ using the Sage admcycles package, which produces divisors with polynomial coefficients using Sage's built in polynomial ring class. Using non linear programming to search for (local) minima, subject to the inequalities satisfied by an F-divisor, we aim to look for a negative top self intersection. If the polynomial is always positive, then one obtains support for the conjecture, and if it is negative, it follows that the conjecture is false.

For example, for $n = 5$ the top self intersection is given by a homogeneous polynomial of degree 2 in 26 variables. A number of trials using MATLAB in the known cases $n = 5$ and $n = 6$, we have found a variety of (approximate) local minima, some near zero, some bigger, depending on the random starting point. For $n = 7$, the first unknown case, one obtains a homogeneous polynomial of degree 5 in 120 variables, comprised of 88803 monomials. This polynomial is too big for MATLAB to work with. In this case, the approach is to decrease the number of monomials, and hence the complexity of the polynomial, by assuming the divisor of interest contracts certain collections of F-curves. The more F-curves that the divisor kills, the simpler the polynomial.

Choosing which F-curves will be contracted is itself an interesting problem. On the one hand, if the wrong F-curves are chosen, one risks that the divisor will be nef via [Gib24]. Our group has written code to generate large collections of curves that avoid such a scenario. For instance, this code produces 301 F-curves for $n = 7$, and 966 for $n = 8$. Although we cannot guarantee that these are maximal collections, we have used some heuristics and tree-search algorithms aimed at maximization to obtain these lists. On the other hand, if the divisor is assumed to contract too many F-curves, the corresponding polynomial will be trivial. Given that the Picard numbers are 121 for $n = 7$, and 248 for $n = 8$, if the divisor contracts all curves in these "maximal" families, it will be trivial.

After experimenting with the known cases, we have zeroed in on a couple of independent families of F-curves that are small enough, and for which a divisor that contracts them is not obviously nef. For instance, one family consists of $\binom{n}{2}$ curves, and another has $2n - 3$ curves. We are currently studying the positivity of the associated top self intersection of F-divisors that contract these collections.

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3.4 Connectedness of Type A degeneracy loci

Project Leader: Nathan Pflueger

3.4.1 Overview of the problem

A famous theorem on degeneracy loci is the following connectedness theorem of Fulton and Lazarsfeld [FL81]. It concerns the following situation: $\sigma:\ E \to F$ is a map of vector bundles on a connected variety X, and we consider the degeneracy loci $D_k(\sigma) \subseteq X$ where σ has rank at most k. The theorem states that if $\mathrm{Hom}(E, F)$ is an *ample vector bundle*, then $D_k(\sigma)$ is connected provided that its expected dimension is positive. This theorem has an important application to moduli of linear series on algebraic curves, which motivated its original proof: for a general curve C, the Brill–Noether locus $W^r_d(C)$ is connected when it is positive-dimensional.

This theorem has a natural generalization, which appears, surprisingly, to be open. In addition to a map $\sigma : E \to F$, one also adds the data of flags of subbundles of E and quotients of F:

$$E_1 \subset E_2 \subset \cdots \subset E_r = E \xrightarrow{\sigma} F_s \twoheadrightarrow F_{s-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1.$$

and imposes bounds on the ranks of all maps $E_i \to F_j$. The data of these ranks are encoded by a permutation w, and one obtains a degeneracy locus $D_w(\sigma)$. The geometry of these degeneracy loci is intimately linked to Schubert polynomial of w [Ful92], and recent work has found similar results relating the K-theory and motivic classes of such degeneracy loci to Grothendieck polynomials [ACT22a, ACT22b]. The problem is:

Question 1. Assume that $\mathrm{Hom}(E, F)$ is ample, and the expected dimension of $D_w(\sigma)$ is positive. Are there suitable hypotheses on the permutation w under which $D_w(\sigma)$ is connected?

Some geometric motivation comes from [Pfl21], where these degeneracy loci to study a generalization $W^\Pi(C, p, q)$ of the classical Brill–Noether loci to curves with two marked points. Here Π is a "dot pattern," which is essentially the same as a permutation. That paper establishes several results analogous to the classical theory of $W^r_d(C)$, but not a connectedness theorem. The question above may be able to fill this gap.

Question 2. Are the loci $W^\Pi(C, p, q)$ connected for general twice-marked curves (C, p, q), under suitable hypotheses on Π?

3.4.2 Progress during the workshop

The project group mastered the original argument of Fulton and Lazarsfeld, and found appropriate modifications to it that succeeded in proving connectedness of $D_w(\sigma)$ for several specific examples of w, subject to the expected ampleness and dimension hypotheses. The modifications required nontrivial geometric and combinatorial insight, for which the group's mixture of backgrounds was essential. After working several examples, the group concluded that *Grassmannian permutations* where the most natural class of permutations to study first. They succeeded in formulating a bound C_w, combinatorial in nature and easy to compute, such that $\mathrm{Hom}(E, F)$ ample and $\dim X > C_w$ implies that $D_w(\sigma)$ is connected. In most of specific examples considered, $C_w = \ell(w)$, the length of w, which is the optimal such bound. Further work is needed to classify those Grassmannian permutations for which $C_w = \ell(w)$, improve the bound C_w, or extend beyond Grassmannian permutations. The group intends to continue their collaborate after the workshop and write their results for publication.

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3.5 Piecewise polynomials on moduli of curves

Project Leader: Aaron Pixton

The moduli space of stable curves of genus g with n marked points, $\overline{\mathcal{M}}_{g,n}$, admits a stratification by topological type, called the *boundary stratification*. In this stratification, each stratum corresponds to a stable graph Γ, i.e. the dual graph of a stable curve. One way to encode the combinatorial data of this stratification is via its dual cone complex, the moduli of tropical curves $\mathcal{M}_{g,n}^{\mathrm{trop}} = \Sigma_{\overline{\mathcal{M}}_{g,n}}$, an analogous object to the polyhedral fans appearing in toric geometry. This complex $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ has one ray for each boundary divisor, one 2-dimensional cone for each codimension 2 boundary stratum, and so on.

A *piecewise polynomial* on $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ is a continuous function that can be given by choosing a (finite polyhedral) subdivision of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ and picking one polynomial (over $\mathbb{Q}$) on each cone in the subdivision. There is a correspondence between (certain) subdivisions of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ and spaces $\mathcal{M}$ constructed by repeatedly blowing up $\overline{\mathcal{M}}_{g,n}$ along boundary strata. It turns out that a piecewise polynomial defined on such a subdivision can then be interpreted as giving a class in the Chow ring (or cohomology) of $\mathcal{M}$. This procedure defines a ring homomorphism (see e.g. Molcho-Pandharipande-Schmitt, 2023)

$$\phi : \mathsf{PP}(\Sigma_{\overline{\mathcal{M}}_{g,n}}) \to \varinjlim_{\mathcal{M}} \mathsf{CH}^*(\mathcal{M}),$$

where the left side is the algebra of piecewise polynomials on $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ and the right side is the direct limit of all the Chow rings of such $\mathcal{M}$ (under pullback maps).

We considered three problems about piecewise polynomials on (subdivisions of) $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ during the workshop. The first was to try to understand the "easy relations" between piecewise polynomial classes: there are piecewise polynomials that lie in the kernel of ϕ for simple reasons, essentially dimensional vanishing. We wrote out some explicit examples of this in the case of $\overline{\mathcal{M}}_{1,3}$ and speculated about setting up a filtration of $\mathsf{PP}(\Sigma_{\overline{\mathcal{M}}_{g,n}})$ encoding some of these relations.

One source of interesting elements in $\mathsf{PP}(\Sigma_{\overline{\mathcal{M}}_{g,n}})$ is a formula for the logarithmic double ramification cycle $\mathsf{LogDR}_g(A)$ (Holmes-Molcho-Pandharipande-Pixton-Schmitt, 2022), where A is an n-tuple of integers with sum 0. The most complicated part of this formula is simply describing the subdivision of $\Sigma_{\overline{\mathcal{M}}_{g,n}}$ that it uses. This requires some graph theory involving stability conditions for compactified universal Jacobians (Kass-Pagani, 2019). In the simplest nontrivial case, $\mathsf{LogDR}_g(2, -2)$, the subdivision can be understood quite explicitly using 2-edge cuts. The second problem was to see if we could understand what is going on with the subdivision for $\mathsf{LogDR}_g(3, -3)$. We thought a bit about what this subdivision looks like on some specific cones.

The third problem we considered was posed during the workshop - the question is whether one can write an "efficient" set of generators for the polynomial algebra $\mathsf{P}(\Sigma_{\overline{\mathcal{M}}_{g,n}})$ (i.e. the piecewise polynomials in which no subdivision is used). There is a tempting candidate here, corresponding to the classes of pure boundary strata, and we checked that these suffice in various small cases. However, we found an example of a polynomial on $\Sigma_{\overline{\mathcal{M}}_{10}}$ that *cannot* be written as a polynomial in these proposed generators.

3.6 Strata of canonical divisors on algebraic curves

Project Leader: Nicola Tarasca

Spaces of differentials on curves have witnessed an explosion of interest in recent years, motivated by recent developments in the study of limits of differentials on nodal curves [FP2, BCG$^+$1, BCG$^+$3, BCG$^+$2]. By imposing conditions on zero and pole orders, one obtains a natural *stratification* of spaces of differentials on curves. The strata are related to *Witten's r-spin cycles* [PPZ, CJRS] and generalize the *double ramification cycle*, incarnating fundamental aspects of Gromov-Witten theory [JPPZ]. The strata are in fact the projection on moduli spaces of curves of *incidence varieties* living in projectivized Hodge bundles. The proposed research project consists of a direct approach to the study of the cohomology classes of the incidence varieties.

To describe the project, start with the k-th Hodge bundle $\mathbb{E}^k_{g,n}$. This is the vector bundle of stable k-differentials over the moduli space $\overline{\mathcal{M}}_{g,n}$. It is defined as $\mathbb{E}^k_{g,n} := \pi_* \left(\omega_\pi^{\otimes k}\right)$, where $\pi\colon \mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the universal curve with relative dualizing sheaf ω_π. Let $\mathbb{PE}^k_{g,n}$ be the projectivization of $\mathbb{E}^k_{g,n}$. A point of $\mathbb{PE}^k_{g,n}$ consists of a stable n-pointed genus g curve together with the class of a nonzero stable k-differential modulo scaling by units. The space $\mathbb{PE}^k_{g,n}$ compactifies the moduli space of k-canonical divisors on smooth curves.

Given an n-tuple of integers $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ such that $|\mathbf{m}| = k(2g-2)$, one defines the *incidence variety* $\mathbb{H}_{g,\mathbf{m}} \subset \mathbb{PE}^k_{g,n}$ as

$$\mathbb{H}_{g,\mathbf{m}} := \left\{ (C, P_1, \ldots, P_n, \mu) \in \mathbb{PE}^k_{g,n} \,\middle|\, C \text{ is smooth and } \operatorname{div}(\mu) = \sum_{i=1}^{n} m_i\, P_i \right\}.$$

The closure of the incidence varieties in $\mathbb{PE}^k_{g,n}$ has been studied by Bainbridge, Chen, Gendron, Grushevsky, and Möller [BCG$^+$1, BCG$^+$3, BCG$^+$2]. A recursive description of the incidence variety classes on $\mathbb{PE}^k_{g,n}$ has been given in Sauvaget [Sau], but no closed formula is known for these classes.

Specific Goal 1. Compute a closed formula for the incidence variety classes on $\mathbb{PE}^k_{g,n}$.

A preliminary case of Specific Goal 1 is solved in my work with Iulia Gheorghita. Specifically, we have computed a closed formula for the *restriction* of the incidence varieties over the *moduli space* $\mathcal{M}^{\mathrm{rt}}_{g,n}$ *of curves with rational tails* in the holomorphic case [GT]. Our resulting formula is expressed as a sum over decorated stable graphs dual to the boundary strata in $\mathcal{M}^{\mathrm{rt}}_{g,n}$ with coefficients enumerating appropriate weightings of decorated stable graphs.

The solution to Specific Goal 1 could help to make progress toward some open problems on the cohomology of moduli spaces of curves [Pan]. The study of cohomological classes on moduli spaces of curves has been pioneered by Mumford [Mum] and Faber [Fab]. While the whole cohomology is practically unwieldy, a smaller ring, called the *tautological ring*, captures most of the desired features [FP1].

Basic questions about the tautological ring remain open. There is an explicit set of generators for the tautological ring, indexed by decorated stable graphs, but the set of relations between the generators is still unknown. *Some* relations are known, but it is unclear whether *all* the necessary relations are known. There are currently two competing conjectures [Fab, Pix], with one of them known to be false in some cases [PT, Pet].

The solution to Specific Goal 1 provides a new source of relations. Indeed, the proposed strategy produces a formula for classes of strata of k-differentials $\mathcal{H}_{g,\mathbf{m}}$ on $\overline{\mathcal{M}}_{g,n}$ which naturally vanishes in the tautological ring when $|\mathbf{m}| > k(2g-2)$, thus yielding a relation.

Specific Goal 2. Determine the classes of the empty strata $\mathcal{H}_{g,\mathbf{m}}$ in $\overline{\mathcal{M}}_{g,n}$, and verify whether they contribute new tautological relations.

Updates from the activities at Banff

During the week at Banff, my group analyzed the recursive structure of incidence variety classes beyond the locus of curves with rational tails. We performed various computations and investigated what properties could be used to extend the classes over the larger locus of curves of compact type.

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3.7 Curves in $(\mathbb{P}^1)^n$ of multidegree $(1, \dots 1)$

Project Leaders: Rohini Ramadas and Rob Silversmith

The goal of this project is to understand the moduli spaces $\mathcal{X}_n$ parametrizing rational curves in $(\mathbb{P}^1)^n$ of (multi-)degree $(1, 1, \dots, 1)$. More precisely we let $\mathcal{X}_n$ denote the Kontsevich space of stable maps

$$\overline{M}_{0,0}((\mathbb{P}^1)^n, (1, \dots, 1)).$$

Then $\mathcal{X}_n$ has several properties that hint at interesting angles for studying it:

- $\mathcal{X}_n$ is irreducible and smooth of dimension $3n - 3$.
- $\mathcal{X}_n$ admits a system of flat tautological morphisms $\mu : \mathcal{X}_n \to \mathcal{X}_{n'}$ (or precisely, $\mu_S : \mathcal{X}_n \to \mathcal{X}_{|S|}$ for each nonempty $S \subseteq [n]$), analogous to the system of forgetful morphisms $\overline{M}_{0,n} \to \overline{M}_{0,n'}$.

- $\mathcal{X}_n$ has a boundary stratification by smooth divisors with simple normal crossings. The (locally closed) strata are precisely the orbits of the natural $\mathrm{PGL}(2)^n$-action. The open stratum is isomorphic to $\mathrm{PGL}(2)^{n-1}$ and $\mathcal{X}_n$ is a wonderful compactification (i.e. an equivariant compactification with smooth orbit closures). The closed strata are products of smaller-dimensional spaces $\mathcal{X}_{n',m}$ (stable maps to $(\mathbb{P}^1)^{n'}$ with m marked points). Unlike for $\overline{M}_{0,n}$, the stratification doesn't go all the way down to dimension zero — all minimal strata have dimension n.

Example. $\mathcal{X}_2 \cong \mathbb{P}^3$, and its boundary is a smooth quadric surface.

In [1], *cross-ratio degrees* are expressed as intersection numbers on $\mathcal{X}_n$. One motivation behind studying $\mathcal{X}_n$ (in particular its intersection theory) is to get new combinatorial information about cross-ratio degrees. Another motivation is that there seem to be analogies worth investigating with $\overline{M}_{0,n}$ as well as with Grassmannians. (Degree-$(1,\dots,1)$ curves in $(\mathbb{P}^1)^n$ are not as nice as lines in $\mathbb{P}^n$ but not too far off.)

3.7.1 Preliminary questions

We planned to begin by addressing some of the following questions.

1. Describe fibers of $\mu_{[n-1]} \times \mu_{\{1,n\}} : \mathcal{X}_n \to \mathcal{X}_{n-1} \times X_2 \cong \mathcal{X}_{n-1} \times \mathbb{P}^3$. Is this map an iterated blowup? Similarly for the map $\prod_{i=2}^n \mu_{\{1,i\}} : \mathcal{X}_n \to \mathcal{X}_2^{n-1} \cong (\mathbb{P}^3)^{n-1}$. E.g. is $\mathcal{X}_3$ an iterated blow-up of $\mathbb{P}^3 \times \mathbb{P}^3$?

2. Probably using the above, how can we understand the intersection theory of $\mathcal{X}_n$? Are there natural generators for the Chow ring or Chow groups? Are there nice combinatorial rules for multiplying them? (Probably all Chow classes are tautological in the sense of Oprea, this may follow from Oprea's result, it's not immediately clear.)

3. What does the S_n action on $\mathcal{X}_n$ look like? What is $\mathrm{Pic}(\mathcal{X}_n)$ as an S_n representation?

4. What is the boundary stratification of $\mathcal{X}_n$?

5. Does $\mathcal{X}_n$ have interesting real/positive structure?

6. Does the toric structure of $(\mathbb{P}^1)^n$ play a large role? For example, what is the relationship between $\mathcal{X}_n$ and the (birational) space of log stable maps?

3.7.2 Summary of group discussions at BIRS

During the week at BIRS, we addressed some of the above questions, as well as a few others:

1. We outlined a factorization of the map $\prod_{i=2}^n \mu_{\{1,i\}} : \mathcal{X}_n \to \mathcal{X}_2^{n-1} \cong (\mathbb{P}^3)^{n-1}$ as an iterated blow-up of $(\mathbb{P}^3)^{n-1}$. We did this in some detail for $n = 3, 4$.

2. We used the above description to guess a description of $\mathrm{Pic}(\mathcal{X}_n)$ over $\mathbb{Z}$ and over $\mathbb{Q}$: it is freely generated over $\mathbb{Q}$ by the $2^{n-1} - 1$ boundary divisors, but is not generated over $\mathbb{Z}$ by boundary. The boundary divisors generate a lattice of index 2^{n-1}. We verified in some examples that $\mathrm{Pic}(\mathcal{X}_{n,m})$ is not generated over $\mathbb{Q}$ by boundary once $m > 0$.

3. We did some basic intersection calculations in the Chow ring, such as computing the pullback of a boundary divisor along the projection maps μ_S, and the pullback of the class of a point in $(\mathbb{P}^1)^2$ along the evaluation map $\mathcal{X}_{2,1} \to (\mathbb{P}^1)^2$.

4. We studied the boundary complex of $\mathcal{X}_{n,m}$, a simplicial complex of dimension $3n - 4 + m$. We first showed that the boundary complex has Euler characteristic 1 using sign-reversing involutions, then refined this result by showing that in fact the boundary complex is contractible for all $n \geq 1$ and $m \geq 0$.

5. We talked through the $(\mathrm{PGL}_2)^n$ action on $\mathcal{X}_n$, and sketched a proof that $\mathcal{X}_n$ is a wonderful compactification of a $(\mathrm{PGL}_2)^{n-1}$ as a homogeneous $(\mathrm{PGL}_2)^n$-space. We classified the Borel-fixed points and Borel-invariant curves, and the torus-fixed points.

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3.8 Constructing fine compactified Jacobians

Project Leader: Orsola Tommasi

Many geometric problems in the theory of algebraic curves deal with pairs (C, L) where C is a curve and L is a line bundle on C. When the curve C is a smooth curve of genus g, this leads to the study of its Jacobian, the g-dimensional algebraic variety parametrizing all line bundles of a fixed degree d. Its construction behaves well in families and leads to the degree d *universal Jacobian* over the moduli space $\mathcal{M}_g$ of smooth curves of genus g. However, as soon as the curve over which one works has singularities, the Jacobian, defined as the moduli space of line bundles, is no longer a projective variety. In the attempt of compactifying it, pathological behaviour appears.

In this project, we deal with nodal curves. Compactifying their Jacobian involves having to add degenerations of line bundles (torsion-free coherent sheaves of rank 1). These sheaves will not be locally free at some of the nodes of the curve, which we will call the *singular locus* of the sheaf. The combinatorial information needed to encode such a torsion-free sheaf F consists of:

- the dual graph Γ of C;
- the connected subgraph G of Γ obtained by removing the edges of the dual graph that correspond to the singular locus of the sheaf;
- the multidegree $D = (d_v)_{v\in V(\Gamma)}$. The sheaf F is the push-forward of a line bundle at the partial normalization of C at the nodes in the singular locus of F; each d_v represents the degree of the restriction of this line bundle to the preimage of the component of C corresponding to the vertex v.

Each choice of a pair (G, D) gives a locally closed stratum in the moduli space of rank 1 torsion-free sheaves on C. To construct a well-behaved compactification of the Jacobian we have to glue together strata in such a way that their union is proper (e.g. projective, compact) and at the same time open in the moduli space of torsion-free sheaves.

In joint work with Nicola Pagani we found necessary and sufficient conditions on how to choose the strata, involving the action of the twister group (or chip-firing group) of the graph Γ and of its subgraphs on the multidegrees D. However, a classification of these compactified Jacobians has been obtained so far only in few cases.

The aim of the project was to work out some more cases, in the pursuit of new examples of compactified Jacobians. We started by looking at single nodal curves. We reviewed the results for curves of genus 1 and developed new ways to visualize and rephrase the known constructions. Afterwards we considered the compactification induced by working with the so-called break divisors of the dual graph. Furthermore, a participant implemented an algorithm that should allow to look for examples of compactified Jacobians for a given graph of arbitrary genus.

4 Other activities

On Monday evening, after admiring the Elk visiting the Banff Centre that evening, we broke into groups mixed according to mathematical background to discuss similarities and differences between the mathematical communities present at the workshop, particularly related to mathematical communication within geometry and combinatorics. By remixing the groups halfway through the hour, we ensured that the majority of the participants had had a substantive conversation with most other participants by the end of the first day. This discussion served as an effective icebreaker for the conference; we observed a high level of mixing between participants at meals and coffee breaks across the week.

Wednesday afternoon saw most of the participants take part in the informal hike to Hoodoos.

In the evening we got to briefly sample the concert *In a Landscape: Classical Music in the Wild* (thanks to the concert organizers for the loan of headsets) before breaking into groups mixing career stages for a mentoring session. The demographics of the workshop meant that many participants were either supervising their first PhD student or PhD students themselves, so much of the conversation was about what made a good PhD student/supervisor relationship.

A major aim of this workshop was to bring together two different communities: algebraic geometers working on the moduli space of curves, and combinatorialists with potentially useful skills. The last session on Thursday was devoted to discussing potential future interactions, and which forms of activity would be best to deepen these connections. Many participants commented that they would love to attend another workshop structured identically to COMOC; others proposed summer schools and conferences based on exposing geometers to combinatorial techniques (or vice versa).