Nonlinear Water Waves: Rigorous Analysis and Scientific Computing

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1 Overview

Why study water waves? Water waves (that is, waves on the surface of a fluid, or the interface between different fluids) have been seen by everyone, and are naturally physically fascinating. Furthermore, the governing equations are also mathematically fascinating, having apparently every possible complication. A rigorous theory of their solutions is extremely complex due not only to the fact that the water-wave problem is a classical free-boundary problem (where the problem domain, specifically the water surface, is one of the unknowns), but also because the boundary conditions (and, in some cases, the equations) are strongly nonlinear. Nevertheless, the last twenty years have seen rapid progress in analysis and numerical simulation of surface waves, and these developments have been accompanied by a series of highly successful programmes at Isaac Newton Institute in Cambridge (2014, 2017, 2022), Mathematical Sciences Research Institute, Berkeley (2021), Fields Institute (2021) and workshops, notably at Mathematics Research Institute Oberwolfach (2015, 2019), Erwin Schrödinger International Institute for Mathematics and Physics, Vienna (2017).

The organisers are pleased to report the continuing increase in the rate of progress in the mathematical theory of water waves, particularly in interdisciplinary areas (as evidenced by the launch of the new interdisciplinary journal *Water Waves* by Springer in 2019) and in terms of fresh blood interest (as evidenced by the recent explosion in the numbers of highly talented early career researchers). It thus appeared timely to convene a further workshop in Banff to review the state of the art (including in particular a number of recent unexpected breakthroughs using methods from apparently unconnected areas of mathematics). The workshop distinguished itself from other events by concentrating on the interaction between researchers with expertise on rigorous analysis and experts on scientific computing for the exact hydrodynamic equations, bridging the divide between Pure and Applied Mathematics.

There has been a remarkable explosion of activity in the six years since the previous Banff workshop on waves in 2016, from which we selected five themes either as venerable topics which have recently become amenable to innovative new methods, or as exciting emerging areas yielding results for the first time. The themes were

- Water waves with vorticity,
- Initial-value problems and instabilities,
- Quasiperiodic waves and normal-form theory,

- Waves in domains with complex geometry, wave-structure interaction and additional surface effects,
- Waves-ice interaction, hydroelastic waves.

An organic discussion of other recent breakthroughs was included as a separate item. The objectives of the workshop were: to advance research across all these themes; to enable a transfer of knowledge from experienced to early career researchers; and to facilitate synergy between pure and applied mathematics, in particular rigorous analysis and scientific computing. The aims were achieved by bringing together established experts and early career researchers in mathematical analysis, fluid mechanics and scientific computation, and their success will be measured by monitoring the scientific output of the participants in general, and in particular the citation and download data for a special issue of *Water Waves* associated with the workshop.

2 Survey talks

An hour-long survey talk (including one virtual presentation) was given on each of the workshop themes.

- David Lannes (Université de Bordeaux, France), *Some recent results on wave structure interactions*

  In this virtual talk, Prof. Lannes described recent results related to the interactions of ocean waves with floating objects; the problems were motivated in particular by application to marine renewable energies. Depending on the physical model used to describe the waves (water waves or asymptotic models such as nonlinear shallow water or Boussinesq equations), but also on the dimension, and on the geometry of the object and its motion (fixed, constrained or freely floating), different new mathematical questions (corner singularities, initial boundary value problems, free boundary problems with nonkinematic boundary conditions, etc.) were discussed.

- Emilian Părău (University of East Anglia, UK), *Hydroelastic waves and other wave-ice interactions*

  In polar regions, floating ice on frozen lakes is often used as a road to access isolated communities. Under certain conditions, the ice can be modeled as a thin plate, and studying hydroelastic waves propagating at the ice-water interface is essential for the safe use of these ice roads. A better understanding of wave-ice interactions in the Marginal Ice Zone (MIZ) is also important in the context of climate change. The MIZ is the area between the open ocean and the continuous ice cover in the Arctic and Antarctic regions. In this talk Prof. Părău provided an overview of recent results in these areas of research, with a focus on the effects of nonlinearity.

- Erik Wahlén (Lund University, Sweden), *Steady periodic water waves with vorticity*

  Historically, most research on water waves has been based on the assumption of irrotational flow, but in the last decades there has been a lot of progress on waves with vorticity. This development is for example useful for modelling wave-current interactions. In his talk Prof. Wahlén gave an overview of recent analytical work on steady (or travelling) waves with vorticity, focusing on the periodic situation. In the two-dimensional setting this included results on large-amplitude gravity waves, where one finds overhanging waves in addition to the classical peaked highest waves which also appear in the irrotational case. Some existence results obtained in the last years for the more challeneging three-dimensional problem were also discussed.

- John Wilkening (UC Berkeley, USA), *Quasiperiodic water waves*

  Prof. Wilkening presented a framework for computing and studying two-dimensional water waves that are quasi-periodic in space and/or time. This means they can be represented as periodic functions on a higher-dimensional torus by evaluating along irrational directions. In the spatially quasi-periodic case, he considered both traveling waves and the general initial value problem. In both cases, the nonlocal Dirichlet-Neumann operator is computed using conformal mapping methods and a quasi-periodic variant of the Hilbert transform. Traveling waves are obtained either as a generalization of the Wilton ripple problem or through bifurcation from large-amplitude periodic waves. In the temporally quasi-periodic case, a shooting method is used to compute standing waves with 3 quasi-periods as well as hybrid traveling-standing waves that return to a spatial translation of their initial condition at a later time. Many examples were given to illustrate the types of behavior that can occur.

- Sijue Wu (University of Michigan, USA), *Well-posedness of the water-wave problems*

  In her talk Prof. Wu surveyed the current results on local and global well-posedness of the water-wave problems.

3 Masterclasses

Three hour-long masterclasses (including one virtual presentation) were given.

- Mats Erhnström (NTNU, Norway), *Nonlocal and fractional order equations in water waves*

  Prof. Ehrnström gave some background and overview on fractional and nonlocal equations appearing either directly in, or associated to, the water wave problem. Originally arising from the nonlocal coupling of the free interface and the fluid bulk in the Euler equations, the interest for nonlocal effects and operators have increased substantially over the last decade or so and this interest can be seen in other areas of PDE as well. The class focussed on some instances of such equations; the differences between the local and nonlocal cases, when present; and techniques and results that are available, or open.

- Mihaela Ifrim (University of Wisconsin, USA), *Low regularity solutions for water wave equations*

  Prof. Ifrim discussed low regularity solutions for the water wave equations in two space dimensions, and both local and global solutions. Such solutions had been proved to exist earlier but in much higher regularity. The goal was to improve these results and prove local and global well-posedness under minimal regularity and decay assumptions for the initial data. One key ingredient here is a newly introduced method called "balanced cubic estimates". Another is the nonlinear vector field Sobolev inequalities, an idea first introduced by Ifrom and Tataru in the context of the Benjamin-Ono equations.

- Benjamin Akers (Air Force Institute of Technology, USA), *On ripples: bifurcations of bimodal waves*

  In the early 1900's several authors presented expansions to approximate ripples with harmonic resonance at Bond numbers where Stokes' expansion is singular. These waves are now called Wilton ripples. Their procedure is effectively a Lyapunov-Schmidt reduction, and has been employed numerous times to asymptotically approximate, numerically compute, and prove existence of these waves. Historically solvability conditions in the expansion are satisfied using corrections to the wave speed and the ratio of the two modes. In the classic expansion, the ratio of the two modes can take only special values. Recently another expansion has been used to study the bifurcation of small amplitude bimodal ripples, which shows that waves bifurcate with almost all ratios of the two modes provided the problem is expanded in Bond number. In his virtual talk Prof. Akers presented the history of both expansions and explained the connection between the two.

4 Research presentations

Twenty-four research presentations, each of thirty minutes length, were given on topics associated with the workshop themes.

- David Andrade (Universidad del Rosario, Colombia), *Numerical simulations of full Euler equations for a moving seabed*

  In his talk David showed how the conformal mapping technique can be used to accurately model surface wave generation due to seabed deformations. The main point was the ability to specify the seabed deformation in the so-called physical domain instead of an ad hoc parametrization in the canonical domain. This allows one to reproduce laboratory conditions accurately and deal with relatively 'rough' and 'fast' seabed deformations. With this technique at hand David revisited the classical laboratory results obtained by Hammack in the 70's and explored the differences between active and passive tsunami generation.

- Ricardo Barros (Loughborough University, UK), *On the coexistence of internal solitary waves of opposite polarity in a three-layer fluid*

  Ricardo considered a three-fluid system confined between two rigid walls. It is well known that the Gardner equation, when adopted to describe long waves in this physical system, is able to predict the coexistence of internal solitary waves of opposite polarity, for some physical parameters. This is so because, across the parameter space, the cubic nonlinearity coefficient for the Gardner equation may change sign. Ricardo tested the validity of these results by analysing the predictions of the strongly, and fully, nonlinear theories.

- Mark Blyth (University of East Anglia, UK), *Stability of waves on fluid of infinite depth with constant vorticity*

  Mark examined the stability of two-dimensional waves on the surface of inviscid fluid of infinite depth in the presence of a constant vorticity shear current. The basic state solution corresponds to a periodic travelling wave that is moving at constant speed. The branch of periodic travelling waves is remarkable in that it can be described by an exact solution (Hur & Wheeler 2020). Mark's concern was with the linear stability of these travelling waves. He studied this first for small amplitude waves, identifying resonant triad interactions leading to instability for non-zero wave amplitude. Stability for arbitrary amplitude waves was determined numerically using a Floquet-Fourier-Hill approach and a collocation method. The effect of weak gravity was also considered.

- Wooyoung Choi (New Jersey Institute of Technology, USA), *Modulation of short surface waves interacting with long internal waves*

  Wooyoung studied the modulation of short surface gravity waves interacting with long internal waves in a two-layer system. A second-order nonlinear system derived from the two-layer Euler equations was studied numerically and it was shown that the modulation is much pronounced under a near-resonance condition. The results were compared with laboratory experiments and possible reduced models were discussed.

- Didier Clamond (Université Côte d'Azur, France), *On the recovery of surface waves from bottom pressure*

  Didier presented recent results on surface wave determination from measurements of the pressure at the seabed. In particular, he discussed the effects of surface pressure (wind, surface tension, etc.) and of noise in the measurements.

- Ryan Creedon (Brown University, USA), *Transverse instability of Stokes waves*

  In 1981, McLean discovered via numerical methods that Stokes waves, even those of small amplitude, are unstable with respect to transverse perturbations of the initial data. A proof of this fact has been missing ever since. In his talk, Ryan presented the first ever proof of the existence of transverse instabilities of Stokes waves. In particular, it was shown that the linearization of the full water wave equations about the Stokes waves admits solutions that grow exponentially in time. The proof is joint work with Huy Nguyen at the University of Maryland and Walter Strauss at Brown University.

- Bernard Deconick (University of Washington, USA), *The instabilities of near-extreme surface waves*

  Bernard provided an overview of relevant results (analytical, asymptotic, computational) in the literature, both classical and more recent on the instabilities of water waves. He especially emphasised results for steep waves, near the extreme 120 degree wave.

- Mark Groves (Saarland University, Germany), *Fully localised three-dimensional solitary water waves on Beltrami flows with strong surface tension*

  Fully localised three-dimensional solitary waves are steady water waves which are evanescent in every horizontal direction. In this talk Mark presented an existence theory for such waves under the assumptions that the relative vorticity and velocity fields are parallel ('Beltrami flows', which include the special case of irrotational flows), that the free surface of the water takes the form $\{z = \eta(x, y)\}$, and that the influence of surface tension is sufficiently strong. The governing equations were formulated as a single equation for η, which was then shown by an argument of Lyapunov-Schmidt type to be locally equivalent to a perturbation of the KP-I equation

$$\partial_x^2\left(\partial_x^2\zeta - \zeta - 3\zeta^2\right) - \partial_y^2\zeta = 0. \qquad (\star)$$

It has recently been shown by Liu, Wei and Yang that $(\star)$ has an infinite family $\{\zeta_k\}$ of symmetric lump solutions of the form

$$\zeta_k(x,y) = -2\partial_x^2 \log \tau_k(x,y), \qquad k = 1, 2, \ldots,$$

where τ_k is a polynomial of degree $k(k+1)$ given by an explicit formula. They also showed that ζ_1 and ζ_2 are nongenerate and announce that the same is true for ζ_k, $k \geq 3$ (details to be published in a later paper). Using these results, one can apply a suitable variant of the implicit-function theorem to construct fully localised three-dimensional solitary waves which are perturbations of scalings of the Liu-Wei-Yang lump solutions to the KP-I equation.

- Bastian Hilder (TU Munich, Germany), *Global bifurcation of doubly periodic water waves in Beltrami flows*

  In this talk, Bastian presented a global bifurcation result for doubly periodic gravity-capillary water waves with vorticity. Specifically, he considered the case that the velocity field is a Beltrami field; that is, the velocity and vorticity fields are collinear. The result is based on a reformulation of the problem as identity + compact and analytic global bifurcation theory. This is joint work with Giang To and Erik Wahln (Lund).

- Dan Hill (Saarland University, Germany), *Building a round peg: developing a framework for waves in cylindrical geometries*

  There has been recent interest in the study of surface waves in cylindrical domains, with a particular focus on axisymmetric solitary waves and periodic waves. Examples include waves travelling along the length of a horizontal cylinder of fluid, as well as axisymmetric waves (i.e. targets or spots) on the upper surface of a vertical cylinder of fluid. In each case, the rotational symmetry reduces the problem to a quasi- two-dimensional PDE with spatially-dependent terms and continuity conditions at the origin. While restricting to axisymmetric solutions should represent a simplification of the full three-dimensional problem, much of the analytic framework required to study such solutions is not well-established in polar coordinates. In this talk Dan explored an analytic framework for radial functions, utilising non-autonomous differential operators and novel functions spaces, and discussed their application in the context of surface waves. He also discussed vector calculus, flattening transformations and the Stokes stream function, exploring how each topic relates to his framework.

- Vera Hur (University of Illinois, USA), *Stable undular bores: rigorous analysis and validated numerics*

  Vera discussed the global nonlinear asymptotic stability of the traveling front solutions to the Korteweg-de VriesBurgers equation, and other dispersive-dissipative perturbations of the Burgers equation. Earlier works made strong use of the monotonicity of the profile, for relatively weak dispersion effects. Vera exploited the modulation of the translation parameter, establishing a new stability criterion that does not require monotonicity. Instead, a certain Schrodinger operator in one dimension must have exactly one negative eigenvalue, so that a rank-one perturbation of the operator can be made positive definite. Counting the number of bound states of the Schrodinger equation, she found a sufficient condition in terms of the width of a front. She analytically verified that her stability criterion is met for an open set in the parameter regime including all monotone fronts. Numerical experiments, revealing more stable fronts, suggest a computer-assisted proof. This is joint work with Blake Barker, Jared Bronski, and Zhao Yang.

- Delia Ionescu-Kruse (Romanian Academy, Bucharest), *Nonlinear two-dimensional water waves with arbitrary vorticity*

  Delia considered the two-dimensional water-wave problem with a general non-zero vorticity field in a fluid volume with a flat bed and a free surface.The nonlinear equations of motion for the chosen surface and volume variables are expressed with the aid of the Dirichlet-Neumann operator and the Green's function of the Laplace operator in the fluid domain. Moreover, she provided new explicit expressions for both objects. The field of a point vortex and its interaction with the free surface was studied as an example. In the small-amplitude long-wave Boussinesq and KdV regimes, she obtained appropriate systems of coupled equations for the dynamics of the point vortex and the time evolution of the free surface variables. This is joint work with Rossen Ivanov.

- Rosa Maria Vargas Magana (University of Edinburgh, UK), *Lagrangian acceleration as a diagnostic for wave breaking in the nearshore zone*

  Nearshore hydrodynamics involves complex interactions across multiple spatial and temporal scales, driven by various wave processes. Among these, wave breaking and the associated energy dissipation are particularly important. Despite advances in understanding wave breaking, the precise mechanisms that trigger it remain unresolved. This study explored the use of the Lagrangian downward acceleration of fluid particles near the wave crest as a dynamic criterion for identifying wave breaking in shallow waters. The idea of using downward acceleration as an indicator traces back to Longuet-Higgins (1963), who theoretically showed that the acceleration near the crest of a regular wave is -0.5g for the highest wave in deep water. While several threshold values have been proposed through theoretical and experimental investigations, no general consensus has been reached. Using stereo imaging of tracer particles in the surf at Sylt, Germany, coupled with careful data analysis and visual assessment of approximately 100 wave events, Rosa demonstrated that this criterion accurately classifies breaking and non-breaking waves in over 91% of cases. This work is in collaboration with H. Kalisch, M. Buckley, M. Bjrnestad, T. Gronemann, K. Holand, J. Horstmann, M. Streßer, and S. Fromenteau.

- Paul Milewski (Pennsylvania State University, USA), *Resonance of free surface water waves in cylindrical containers*

  Waves sloshing in a container of rectangular cross-section can behave very differently than those in a circular cylinder. This talk covered one aspect of how geometry affects the nonlinear evolution of waves. Nonlinear resonance is a mechanism by which energy is continuously exchanged between a small number of wave modes, and is common to many nonlinear dispersive wave systems. In the context of free-surface gravity waves such as ocean surface waves, nonlinear resonances have been studied extensively over the past 60 years (and were mentioned in the award of the 2021 Nobel Prize in Physics to Klaus Hasselmann) almost always on domains that are large compared to the characteristic wavelength. In this case, the dispersion relation dictates that only quartic (4-wave) resonances may occur. In contrast, nonlinear resonances in confined three-dimensional geometries have received relatively little attention, where, perhaps surprisingly, stronger 3-wave resonances do occur. Paul presented results completely characterizing the configuration and dynamics of resonant triads in cylindrical basins of arbitrary cross sections, demonstrating that these triads are ubiquitous, with a rectangular cross section being an exception where they do not occur.

- Sunao Murashige (Ibaraki University, Japan), *Local flow structure near the separation point of gravity currents*

  The gravity current is the flow of one fluid within another caused by the density difference between the two fluids. Benjamin derived an approximate solution on the assumption that the interface moves in permanent form with constant speed and a corner flow is created near the separation point. This work evaluated the accuracy of Benjamin's approximate solution, improving it using local flow analyses near some critical points of gravity currents. The results show that the secondary singularities at the critical points are essential for gravity currents.

- David Nicholls (University of Chicago, USA), *High-order spectral methods for the computation of Dirichlet-Neumann operators for Laplaces equation with quasiperiodic boundary conditions*

  In this talk David described stable high-order spectral algorithms for the numerical simulation of Dirichlet-Neumann operators (DNO) which arise in boundary value and free boundary problems from a wide variety of applications (e.g., fluid and solid mechanics, electromagnetic and acoustic scattering). More specifically, he considered DNO defined on domains inspired by the simulation of ocean waves subject to quasi-periodic boundary conditions. It was recently shown that the DNO, when perturbed from a flat interface configuration, is parametrically analytic (as a function of deformation height/slope) for profiles of finite smoothness. The method of proof suggests a stable and high-order method of numerical simulation.

- Dag Nilsson (Mid-Sweden University), *Periodically modulated solitary waves of the CH-KP equation*

  The Camassa-Holm-KP equation (CH-KP equation) is a two-dimensional extension of the Camassa-Holm equation, similarly to how the regular KP equation is a two-dimensional extension of the KdV equation. In the talk Dag outlined how to prove existence of periodically modulated waves, i.e, steady solutions which have a solitary wave profile in the x-direction and a periodic profile in the y-direction. This is achieved through reformulating the problem as a dynamical system for a perturbation of the line solitary wave solutions, where the periodic direction takes the role of time, then applying an infinite dimensional version of the Lyapunov centre theorem. The talk is based on a joint work with Douglas Svensson Seth (NTNU) and Yuexun Wang (Lanzhou University).

- Josh Shelton (University of Bath, UK), *The instability and time-evolution of water waves with vorticity and point vortices*

  Recently, many exact solutions have been found for surface waves travelling upon a fluid with vorticity and submerged vortices. The simplest of these, with either constant vorticity or a single point vortex within the periodic domain, possesses the same exact solution for capillary waves originally obtained by Crapper. More complicated exact solutions have vorticity and multiple vortices within the periodic domain. These exact solutions were found via conformal mapping and analytic function theory. However, they are likely to be unstable, so the study of their instability and subsequent time evolution is of interest. In his talk, Josh discussed a potential-flow formulation of this problem, the study of solution stability, and a 1D surface formulation of the 2D problem in which the vortex evolution conditions within the fluid are still satisfied.

- Raphael Stuhlmeier (University of Plymouth, UK), *Resonant wave-wave interactions via phase-plane analysis: an introduction and survey of some recent results*

  The resonant interaction of waves is a significant energy-transfer mechanism in our oceans, and is associated with instabilities such as the Benjamin-Feir (or modulational) instability. Provided there are sufficiently many conserved quantities the resonant interaction equations can be integrated explicitly in terms of elliptic functions. This approach of integration, pursued at least since the 1960s in the context of water waves, leads to the somewhat cumbersome task of classifying the roots of a polynomials which depend on often rather complicated coefficients. It also threatens to obscure the connection between a resonant interaction and the associated (linear) instability, particularly as the latter is typically approached via the assumption of small modal amplitudes. In this talk, Raphael suggested that reducing resonant interaction to two variables (a planar dynamical system) is preferable to reduction to a single variable (integration). By suitable choice of dynamic phase and energy variables, the classification of solutions can be simplified greatly, and it is often possible to identify physically interesting bifurcation parameters. The phase plane allows for a transparent view of the associated dynamics, and a clear role for the linear stability analysis. Moreover, avoiding elliptic functions makes the presentation more pedagogically accessible. Raphael illustrated the above with a variety of examples from water waves and hydrodynamics, aiming to show how new insight can be gained, and new interpretations of classical results found, from the dynamical systems perspective.

- Douglas Svensson Seth (NTNU, Norway), *Asymmetric travelling capillary-gravity waves*

  Periodic travelling waves that solve the capillary-gravity Whitham equation have been fully characterised in the case of small and even waves. This characterisation is complemented by the work presented in this talk dealing with small asymmetric periodic travelling waves. Such asymmetric waves are far more scarce than the even ones and can only be constructed in certain cases for weak surface tension. The method also generalizes in a straightforward way to a class of similar equations for which we either can prove the existence of or non-existence of asymmetric solutions. However, the proof relies on some technical calculations that are different for each equation. Douglas discussed how this can be done for the Babenko equation, which is equivalent to the full water wave problem, to determine the existence of small amplitude capillary-gravity waves.

- Daniel Tataru (UC Berkeley, USA), *Sharp Hadamard local well-posedness, enhanced uniqueness and pointwise continuation criterion for the incompressible free boundary Euler equations*

  Daniel provided a complete local well-posedness theory in H^s based Sobolev spaces for the free boundary incompressible Euler equations with zero surface tension on a connected fluid domain. His well-posedness theory includes: (i) Local well-posedness in the Hadamard sense, i.e., local existence, uniqueness, and the first proof of continuous dependence on the data, all in low regularity Sobolev spaces; (ii) Enhanced uniqueness: the uniqueness result holds at the level of the Lipschitz norm of the velocity and Hölder $\frac{1}{2}$ regularity of the free surface; (iii) Stability bounds: construction of a nonlinear functional which measures, in a suitable sense, the distance between two solutions (even when defined on different domains) and shows that this distance is propagated by the flow; (iv) Energy estimates: refined, essentially scale invariant energy estimates for solutions, relying on a newly constructed family of elliptic estimates; (v) Continuation criterion: the first proof of a sharp continuation criterion at the same level as the Beale-Kato-Majda criterion for the boundaryless case; (vi) A novel proof of the construction of regular solutions. The entire approach is in the Eulerian framework and can be adapted to work in more general fluid domains.

- Samuel Walsh (University of Missouri), *Finite-time self-similar implosion of hollow vortices*

  In this talk, Sam presented some recent results on the finite-time blowup of hollow vortices. These are solutions of the two-dimensional Euler equations with the fluid domain being the complement of finitely many Jordan curves $\Gamma_1, \ldots, \Gamma_M$. The flow is irrotational and incompressible, but with a nonzero circulation around each boundary component. The "vortex core" bounded by each Γ_k is modeled as a bubble of ideal gas: the pressure is constant in space and inversely proportional to the area of the vortex. Sam's results have two parts. First, for all $m > 3$, he constructs a family of solutions taking the form of a near-circular m-fold symmetric hollow vortex that collapses self-similarly into the origin in finite time. This represents an implosion in the sense that the pressure in the vortex core simultaneously diverges to infinity. He also obtains a rigidity result: for $m = 2$ and $m = 3$, the purely circular collapsing vortices are locally unique among all collapsing vortices with uniform velocity at infinity. The second part concerns configurations of multiple hollow vortices. The existence of collapsing configurations of point vortices is classical, and he proves that generically, they can be desingularised to yield a families of hollow vortex configurations that exhibit self-similar finite-time implosion. Specific examples of an imploding trio and quartet of hollow vortices were given. This is joint work with Ming Chen (University of Pittsburgh) and Miles Wheeler (University of Bath).

- Jörg Weber (University of Vienna, Austria), *Axisymmetric water waves connecting to static unduloids*

  Usually steady water waves are constructed by perturbing (laminar) flows with a flat surface. In the setting of axisymmetric capillary water waves, Jörg went in the somewhat opposite direction, starting at static configurations with constant mean curvature surfaces, so-called unduloids. As an interesting interplay between water waves, geometry, and elliptic integrals, he showed rigorously that to any such configuration there connects a global continuum of non-static configurations, which confirms previous numerical observations. Here he allowed for arbitrary vorticity and swirl. This is joint work with Anna-Mariya Otsetova (Aalto) and Erik Wahln (Lund).

- Xinyu Zhao (New Jersey Institut of Technology, USA), *Spatially quasi-periodic water waves*

  Xinyu presented a framework for computing and studying two-dimensional spatially quasi-periodic gravity-capillary water waves of finite depth. Specifically, she adopted a conformal mapping formulation of the water wave equation and represent quasi-periodic water waves by periodic functions on a higher-dimensional torus. She presented numerical examples of traveling quasi-periodic water waves and the time evolution of water waves over quasi-periodic bathymetry. She also discussed an approach to extend this study to three dimensions.

5 Short presentations

There were eleven short viritual presentations of ten minutes length on recent breakthroughs.

- David Ambrose (Drexel University, USA), *The Birkhoff-Rott integral for non-decaying, non-periodic flows*

  The Birkhoff-Rott integral gives the fluid velocity on a vortex sheet, and is the basis of some versions of well-posedness theory for water waves. The Birkhoff-Rott integral has two usual forms, one for flows which decay at horizontal infinity, and one for flows which are horizontally periodic. David gave a new formula for the Birkhoff-Rott integral which unifies and extends these: it is a single formula which applies in both the decaying and periodic cases, and which applies more generally.

- Matt Durey (University of Glasgow, UK), *Resonant triads of gravity waves in confined basins*

  Confining a liquid to a partially filled basin can lead to the spontaneous formation of resonant triads of gravity waves, in which three standing wave modes interact and continuously exchange energy. The existence of these triads depends critically on the dimensions of the container and the depth to which it is filled. Furthermore, the internal resonance induces waves of relatively large magnitude, representing a potential hazard in many industrial and geophysical settings, from the transport of liquid cargo to the flooding of lakes induced by wind-driven seiches. Although recent progress has been made in understanding the onset and evolution of these resonant triads for small-amplitude waves in simple geometries, there remain a number of challenges in the realms of both rigorous analysis and scientific computation. In this talk Matt outlined some of the key open questions in developing a complete understanding of the existence and evolution of resonant triads in confined basins, and their mitigation in industrial settings.

- Anna Geyer (Delft University of Technology, Netherlands), *On the transverse stability of smooth solitary waves in a two-dimensional CamassaHolm equation*

  The Camassa-Holm equation models the unidirectional propagation of waves in shallow water. The stability of its solitary-wave solutions with respect to perturbations in the direction of propagation has been extensively studied. In this short talk Anna focussed on transverse stability. To this end she considered a two-dimensional generalisation of the Camassa-Holm equation and studied the spectrum of the operator which arises after linearisation around the perturbation in suitably weighted spaces. The double eigenvalue related to translational symmetry breaks into a pair of asymptotically stable resonances, and the continuous spectrum is located in the left half-plane. The linear stability of small-amplitude solitary waves with respect to transverse perturbations is established by performing careful resolvent estimates and making use of an asymptotic reduction of CH to KdV. This is joint work with Yue Liu and Dmitry Pelinovsky.

- David Henry (University College Cork, Ireland), *Higher-order integrable models for oceanic internal wave-current interactions*

  In this short talk David derived a higher-order KdV equation (HKdV) as a model to describe the unidirectional propagation of waves on an internal interface separating two fluid layers of varying densities. The model incorporates underlying currents by permitting a sheared current in both fluid layers, and also includes Coriolis forces. He showed that there is an explicit transformation connecting our derived HKdV with integrable equations of a similar type, namely KdV5, Kaup-Kuperschmidt equation, Sawada-Kotera equation, Camassa-Holm and Degasperis-Procesi equations. This is joint work with Rossen Ivanov and Zisis Sakellaris.

- Henrik Kalisch (Bergen University, Norway), *Wave driven circulation*

  Wave action in the surf zone is the main factor in setting up current and circulation patterns, including cross-shore and long-shore currents, exit and entrance flows and vortex motion. These wave-driven currents have a significant impact on sediment transport, beach erosion, as well as the distribution of marine microorganisms and pollutants. Understanding the nature of surfzone circulation patterns is also important from the point of view of beach safety in particular in the context of rip currents.

Henrik employed the Boussinesq Ocean and Surf Zone model (BOSZ) developed by Volker Roeber to explore circulation patterns, focusing particularly on the role of nearshore vortices. More precisely, the emphasis was on understanding how wave parameters such as waveheight, mean period, mean direction as well as environmental conditions such as tidal elevation and bathymetry, influence the nature of the resulting surfzone circulation.

- Jack Keeler (University of East Anglia, UK), *Towards eliminating the nonlinear Kelvin wake*

  Everyone has seen the V-shaped Kelvin wake-pattern visible in the wake of a moving object on the surface of water. These patterns are a rare example of a fluid dynamics phenomena well-known to scientists and layman alike. However, the wake is undesirable for a number of reasons; it can cause erosion to river banks and cause wave-drag, thus reducing the fuel efficiency. Therefore, the design of a moving body that can reduce or even eliminate these waves is important for sustainability. A typical approach is to model the boat by an imposed pressure distribution in the free-surface Bernouilli condition. In this talk Jack showed, using a simple mathematical argument, that by a judicious choice of a pressure distribution, wave-free solutions are possible in the context of a model system; the forced Kadometsev-Petviashvili equation. Strikingly, he showed that these solutions are stable, so they could potentially be visualised in a physical experiment.

- Calin Martin (University of Vienna, Austria), *Azimuthal equatorial flows: exact solutions, stability and wave-breaking results*

  Calin presented some recent results concerning exact solutions to the equations of motion in cylindrical coordinates. Furthermore, he gave some criteria for the stability of the exact solutions as well as wave-breaking results.

- Roberto Ribeiro (Federal University of Paran, Brazil), *The effect of normal electric fields on Stokes drift*

  In periodic wave motion, particles beneath the wave undergo drift in the direction of wave propagation, a phenomenon known as Stokes drift. While extensive research has been conducted on Stokes drift in water wave flows, its counterpart in electrohydrodynamic flows remains relatively unexplored. In this talk Roberto discussed a numerical investigation of Stokes drift beneath periodic traveling irrotational waves in a dielectric fluid under the influence of normal electric fields. Through numerical simulations utilizing conformal mapping, he computed particle trajectories and analysed the resultant Stokes drift behavior beneath periodic traveling waves. The findings indicate that variations in the electric field impact particle velocities while maintaining trajectory shapes. Moreover, the kinetic energy associated with a particle depends on its depth location and is a non-decreasing convex function in a moving frame and constant in a laboratory frame.

- Guido Schneider (University of Stuttgart, Germany), *Notes on the cylindrical KdV approximation*

  The cylindrical KdV equation can be derived as a long wave approximation for the description of radially symmetric waves for a number of 2D dispersive systems. Guido discussed a few aspects related to the validity question of this formal approximation and presented an approximation result for a 2D Boussinesq equation. This is joint work with D. Pelinovsky.

- Shu-Ming Sun (Virginia Polytechnic and State University, USA), *Stability of solitary-wave solutions for generalised abcd-Boussinesq systems*

  Shu-Ming discussed the solitary-wave solutions of the so-called $abcd$-Boussinesq system, which is a model of two equations that can describe the propagation of small-amplitude long waves in both directions in water of finite depth and was derived by Bona-Chen-Saut. If the system is Hamiltonian, where the parameters b and d satisfy $b = d > 0$, small solutions of the system in the energy space are globally defined. Then, a variational approach can be applied to establish the existence and nonlinear stability of the set of solitary-wave solutions for generalised $abcb$-Boussinesq systems. The main idea of the proof is to show that the traveling-wave solutions of the generalised $abcb$-Boussinesq system converge to nontrivial solitary-wave solutions of the generalised Korteweg-de Vries equation. (This is joint work with R. de A. Capistrano-Filho and J. R. Quintero)

- Miles Wheeler (University of Bath), *Overhanging solitary gravity waves*

  Miles constructed gravity water waves with constant vorticity having the approximate form of a disk joined to a strip by a thin neck. This is the first rigorous existence result for such waves, which have been seen in numerics since the 80s and 90s. The method is related to the construction of constant mean curvature surfaces through gluing, and involves combining three explicit solutions to related problems: a disk of fluid in rigid rotation, a linear shear flow in a strip, and a rescaled version of an exceptional domain discovered by Hauswirth, Hlein, and Pacard. This is joint work with Juan Dvila, Manuel del Pino, and Monica Musso.