Representation Theory and Topological Data Analysis

Thomas Brstle (Universit de Sherbrooke and Bishop's University),
Claire Amiot (Universit Grenoble Alpes),
Sergio Estrada (Universidad de Murcia),
Steve Oudot (Inria),
Luis Scoccola (University of Oxford)

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1 Abstract for the workshop

A central problem in data-driven scientific inquiry is how to quantitatively describe the organizational structures intrinsic to large data sets. The field of algebraic topology provides a potential solution via the language of homology, which measures various features in a given topological space - such features being, loosely speaking, holes of different dimensions (e.g. connected components, loops, trapped volumes, etc.). In principle, these features can be located and studied explicitly.

In practice however, fundamental challenges in data analysis, such as the choice of scale or the presence of noise, make it necessary to go beyond the use of numerical summaries on a single topological space.

This need has given rise to the emerging area of topological data analysis, and to its mathematical foundations called persistence theory, whose aim is to define and study homological invariants for parametrized families of topological spaces.

While the one-parameter instance of persistence theory is by now well understood, there are fundamental mathematical and computational challenges associated with the development of its multi-parameter instance.

Recent advances have demonstrated that this new topic can greatly profit from using techniques developed in representation theory, in particular techniques based on homological algebra.

Therefore, the aim of this workshop was to bring together leading researchers as well as emerging scholars from topological data analysis and from representation theory, in order to enhance the growing connections between both areas, in particular, but not limited to, new methods in multi-parameter persistence.

2 Structure of the workshop

The workshop consisted of research talks and group working sessions. A lot of emphasis was put on the working sessions, with the talks serving as introduction and motivation, and generating discussions between participants. Nine talks were given, four in representation theory of finite dimensional algebras, and five on topological data analysis and persistence theory. Talks were accessible to all (on-site and online), while working groups ended up taking place only on-site, due to the importance of being in front of the same board.

The participants represented both areas in equal proportion. The rationale behind the balance between talks and working groups, and the choice of the talk subjects was to foster interactions between the topological data analysis community and the representation theory community. Each working group revolved around one new direction at the interface between the two fields.

3 Talks

- Ulrich Bauer. Topological data analysis and persistent homology: an overview
  I will survey some recent results on theoretical and computational aspects of persistent homology (in one parameter) and its use in topological data analysis. I will illustrate various aspects of persistent homology: its structure, which serves as a topological descriptor, its stability with respect to perturbations of the data, its computation on a large scale, and connections to Morse theory. These aspects will be motivated and illustrated by concrete examples and applications, such as: reconstruction of a shape and its homology from a point cloud, faithful simplification of contours of a real-valued function, existence of unstable minimal surfaces, and identification of recurrent mutations in the evolution of COVID-19.

- Sira Gratz. Introduction to representation theory of finite dimensional algebras

- Woojin Kim. The Generalized Rank Invariant: Mbius invertibility, Discriminating Power, Computation, and Connection to Other Invariants
  Unlike one-parameter persistent homology, the absence of a canonical method for quantifying 'persistence' in multiparameter persistent homology remains a hurdle in its application. One of the best-known quantifications of persistence for multiparameter persistent homology, or more broadly persistence modules over arbitrary posets, is the rank invariant. Recently, the rank invariant has evolved into the generalized rank invariant by naturally extending the domain of the rank invariant to the collection of all connected subposets of the domain poset. This extension enables us to measure persistence

across a broader range of regions in the indexing poset compared to the rank invariant. Additionally, restricting the generalized rank invariant can enhance computational efficiency, albeit with a potential trade-off in discriminating power. This talk overviews various aspects of the generalized rank invariant: Mbius invertibility, discriminating power, computation, and its relation to other invariants of multiparameter persistence modules.

- Dolors Herbera: An approach to relative homological algebra for persistence modules *The aim of this talk is to present some notions of relative homological algebra that are proving to be useful in the developing of the theory of persistence modules. We will follow closely Sections 3, 4 and 5 of the nice paper [BBH], which in turn follows the track of the theory of relative homological algebra developed by Auslander and Soldberg for artin algebras in [AS], and that was extended to more general settings in [DRSS]. Let $\mathcal{A}$ be an abelian category, and fix a class of objects $\mathcal{X}$. Let $\mathcal{F}_{\mathcal{X}}$ denote the class of short exact sequences in $\mathcal{A}$ that remain exact when applying the covariant functor $Hom_{\mathcal{A}}(X, -)$ for any $X \in \mathcal{X}$. Dually, let $\mathcal{F}^{\mathcal{X}}$ denote the class of exact sequences that remain exact when applying the contravariant functor $Hom_{\mathcal{A}}(-, X)$ for any $X \in \mathcal{X}$. Both $\mathcal{F}_{\mathcal{X}}$ and $\mathcal{F}^{\mathcal{X}}$ define exact structures over $\mathcal{A}$, so one can make relative homological algebra with respect to both of them. Basic problems, in this setting, are to determine the relative projective objects and the relative injective objects, whether such classes of relative projectives/injectives are resolving/corresolving, whether there are minimal resolutions/corresolutions, do we have relative homological invariants? can we compute relative global dimensions?. The answer to such questions, in general, is difficult and we will outline solutions in settings that, according to [BBH], are of interest for persistence theory.*
  [AS] Auslander and Solberg, Relative homology and representation theory I: relative homology and homologically finite subcategories. Comm. Alg. 21 (1993), no. 9, 2995-3031.
  [BBH] Blanchette, Brustle, Hanson. Exact Structures for Persistence Modules. arXiv:2308.01790 (2023).
  [DRSS] Dräxler, Reiten, Smalo , and Solberg, with an appendix by B. Keller, Exact categories and vector space categories. Trans. Amer. Math. Soc. 351 (1999), no. 2, 647-682.

- Eric Hanson. Homological invariants of persistence modules
  A common approach to studying multiparameter persistence modules is to introduce some "invariant" to determine the similarity between two given modules. In this mostly expository talk, we discuss recent research which utilizes techniques from (relative) homological algebra to interpret classical examples of invariants and define new invariants. The Hilbert function/dimension vector, barcode, and (generalizations of) the rank invariant serve as our main examples. If time permits, we will also discuss the relationship between homological invariants and poset embeddings. Portions of this talk are based on joint works with Claire Amiot, Benjamin Blanchette, and Thomas Brstle.

- Hvard Bjerkevik and Luis Scoccola. Bottleneck stability in multiparameter persistence
  One-parameter persistence modules decompose into indecomposables of a very simple form, and two interleaved (i.e., "similar") modules allow a nice matching between their sets of indecomposable summands. In multiparameter persistence, not only is there no hope of classifying indecomposables, but simple counterexamples show that there is no reasonable matching between the indecomposable summands of similar modules. However, strong bottleneck stability results can be proven for certain nice families of modules, including, for instance, the projective modules. We will describe two lines of work motivated by these results. In one line, one allows "splitting apart" indecomposable summands before looking for a matching, which gives a notion of similarity of approximate decomposition of general modules. In the second line, one first approximates arbitrary modules algebraically by simpler ones (with a resolution) and then uses a stability result for these simpler modules. We will discuss existing multiparameter stability results as well as open questions; these suggest that the obstacles to proving stronger stability results are similar for the two approaches, despite their apparent differences.

- Baptiste Rognerud. How to compare finite dimensional algebras?
  In this talk we will explore some of the different methods of comparing finite dimensional algebras. We will start with the simplest: isomorphism and Morita equivalence and we will see that they are far too rigid. A weakening of the Morita theorem leads to the notion of tilting module which is the first step

toward derived equivalences. We will see that, for us, this is a much more interesting notion, allowing us to compare algebras and categories that are a priori very different. The concepts presented will be illustrated by many examples and (a few) conjectures.

- Ezra Miller. Homological algebra and sheaf theory for multipersistence
  Persistent homology with multiple parameters can be phrased in more or less equivalent ways in terms of multigraded modules, or sheaves, or functors, or derived categories. All of these descriptions have in common an underlying partially ordered set indexing a family of vector spaces, and this family is interpreted under increasing layers of abstraction. The simplest objects at any level of abstraction are the "indicator" (or "interval", or "spread") objects, which place a single copy of the ground field at every point of an interval in the underlying poset (an intersection of an upset with a downset). Taking the cue from ordinary persistence, where there is just one totally ordered parameter, a large part of multipersistence theory has revolved around relating arbitrary persistent homology modules as closely as possible to indicator objects. To that end, this survey of perspectives from homological algebra and sheaf theory takes a journey starting with relevant definitions of persistence modules and leading to presentations, resolutions, and stratifications in terms of indicator objects. The way is marked by effective data structures, encodings, and finiteness conditions, leading to syzygy theorems and bounds on homological dimensions.

- Raphael Bennett-Tennenhaus. Persistence modules are representations of species
  Filtered poset representations began with work of Kleiner and Nazarova–Roiter, using a process called differentiation. For a grid they appear in work of Bauer–Botnan–Opperman–Steen. For a possibly infinite poset, representations are nothing but persistence modules. This framework can be unified with quiver representations using the notion of a species equipped with commutativity conditions, introduced by Simson. The path algebra of a quiver and the incidence algebra of a poset are both recovered using the tensor algebra. As an example, I will discuss work of Igusa–Rock–Todorov on continuous versions of type A quivers.

4 Working groups: initial goals and outcomes

4.1 Relative homological algebra and algebraic invariants for persistence

Recently, relative homological algebra has been used to define new invariants for persistence modules and clarify relationships between existing ones [2, 7, 8].

Initial goals.

- Beyond numerical invariants: How can we use the maps in (relative) resolutions, and not just (relative) Betti tables?
- Organizing and connecting existing invariants: Can we generalize the framework of [1]? This is to encompass more invariants being proposed, such as [4].
- Stability results for homological invariants.
- Relationships between magnitude and relative homological algebra.

Progress.

Day 1 We discussed: the possible non-existence of covers in infinite posets, the tension between finite posets (which makes relative homological algebra easier) and infinite posets (which we want when we want to think about stability), and going between the finite and infinite perspective, the interaction of covers and interleavings, and other types of invariants obtained from compression. We then identified some questions for further exploration.

Day 2 Big picture: trying to look at persistence modules through the lens of more manageable ones. In specific "easy" cases (restriction/induction/coinduction for finite full subposets, contraction functors for aligned grids in $\mathbb{R}^d$) there are existing tools, and we are looking for the next reasonable generalization. We discussed those existing tools (functors).

There was a suggestion/idea for using the "Jordan type" as a new invariant for TDA. How does this compare to existing invariants? Can we define it for infinite posets? Stability? We plan to discuss this in the next sessions.

Setup. Let P be a poset. Given a finite poset Q, we have adjunctions between $rep(Q)$ and $rep(f(Q))$ for any order preserving map $f : Q \to P$. Suppose $f(Q)$ is a full subposet of P. Suppose restriction to $f(Q)$ plays nicely with approximation theory.

Questions. Do restriction/ induction preserve intervals? How can we pass covers in $f(Q)$ to cover in Q? What class of objects are preserved by induction/restriction? What class of objects are preserved by "theta"?

Day 3 Excursion.

Day 4 Participant explained the Jordan type. We discussed the Jordan type, and explored its potential as an invariant. We tried computing it for small examples, and showed that it is stronger than the generalized rank invariant. We also observed that the set of short exact sequences additive under the Jordan type may not be an interesting exact structure. We also observed that while it is reminiscent of the multirank, some technical details need to be checked to determine whether or not it is stronger.

See Figure 1 for a summary of the progress.

4.2 Approximate decompositions and their stability

This working group is dedicated to the elaboration of a meaningful stability theory for direct-sum decompositions of multiparameter persistence modules, as the usual interleaving and bottleneck distances do not play out nicely with each other as they do in the one-parameter setting. The starting point is a recent paper by Havard Bjerkevik [6], which replaces the usual interleaving distance by surrogates based on prunings and refinement, providing a better control over the respective direct summands of nearby modules. Questions such as defining prunings and refinements summand-wise, computing or approximating the surrogate distances efficiently, or identifying modules with nicely behaved summands in the vicinity of a given module, will be considered.

See Figure 2 for a summary of the progress.

4.3 Computation of lower hook resolutions

Goal. How to compute minimal projective resolutions relative to lower hook modules for 2-parameter modules?

Progress We work with 2-parameter modules. Essentially, we need to compute the standard projective resolution of an associated 4-parameter module. This 4-parameter module is parameterized by the source and target of kernel maps.

We looked at Koszul complexes but decided that it was not the most crucial aspect of the computation there may be a better way to compute things.

See Figure 3 for a summary of the progress.

4.4 Derived categories in persistence

Initial goals. Study connections between continuous cluster category of type A, as considered by Igusa and Todorov, and the extended persistence diagram, as considered by Bauer, Botnan and Fluhr. Perhaps also consider continuous derived category arising in physics of scattering amplitudes [3].

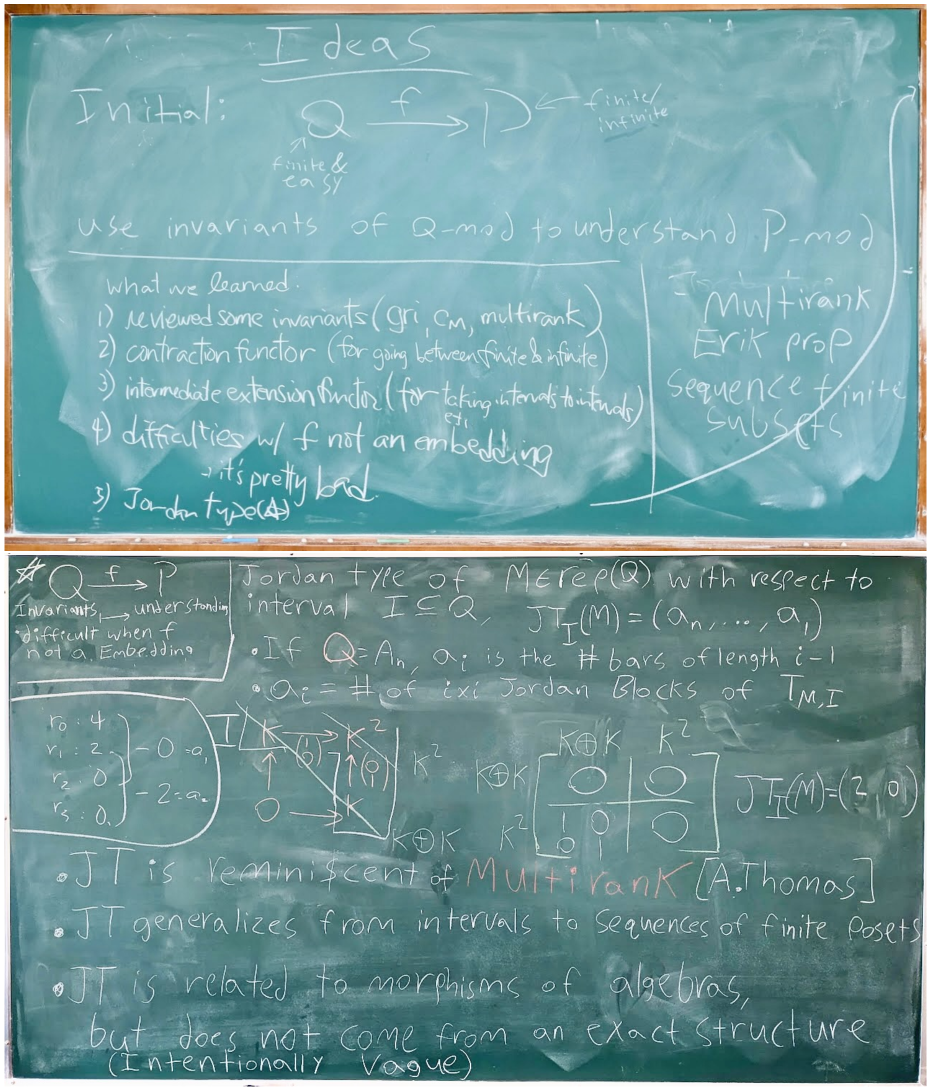

Figure 1: Poster summarizing progress of group 1 (Section 4.1).

Progress.

Day 1 We discussed in detail the extended persistence diagram corresponding to the curve X and its height function $f : X \to \mathbb{R}$ as in Figure 2.3 in Oudot's book [10]. We discussed the level-set zigzags persistence modules corresponding to the same topological space X and height function $f : X \to \mathbb{R}$ and use the algorithm as described in the article [5] to verify the Pyramid Theorem.

Day 2 We discussed further the verification of the pyramid theorem to a particular zig-zag persistence module that is indecomposable and discussed how the obtained pyramid relates to extended persistence. We discussed the definition of cohomological and sequentially continuous persistence modules $V : \mathbb{M}^o \to k - mod$ and discussed if this definition was consistent with the usual definition for cohomological functors and triangulated categories. Gordana also provided a very interesting relation among the approaches in Bauer and Fluhr paper with her article on continuous cluster categories.

Day 3 During the break on Wednesday, we went over [9] and reviewed some of the key concepts related to the construction of certain continuous Frobenius category. On Thursday, a participant gave a presentation about the connection of the aforementioned paper with the article Igusa–Rock–Todorov that improves

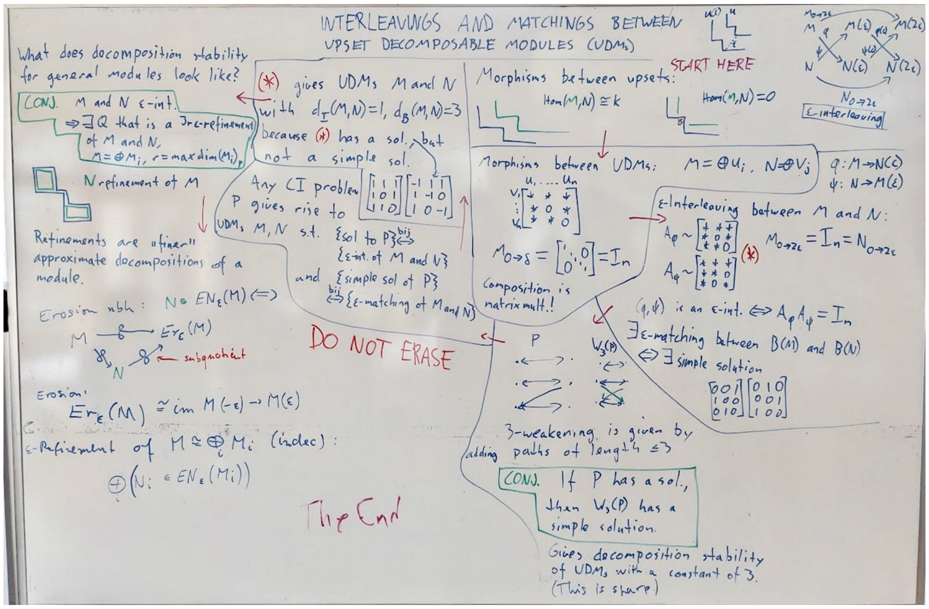

Figure 2: Poster summarizing progress of group 2 (Section 4.2).

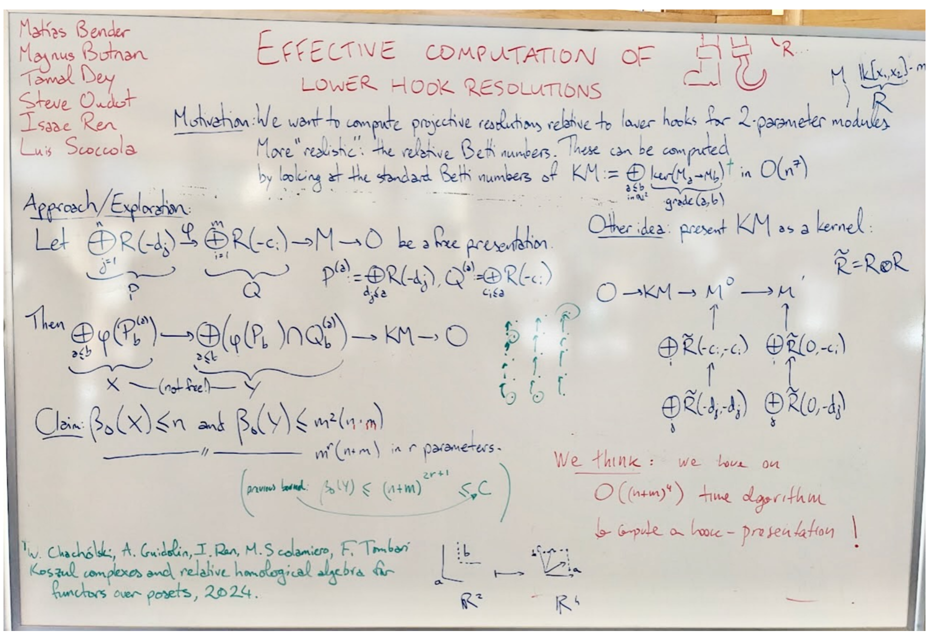

Figure 3: Poster summarizing progress of group 3 (Section 4.3).

the results in the latter paper. They also presented some results on Todorov–Igusa (proceedings Abel Symposium) which was about representations of S^1 and certain class of continuous Frobenius categories. Then we discussed how one can associate points on the strip $\mathbb{M}$ on Bauer–Fluhr's paper to a pair of open subsets of the real line. We also worked on our poster presentation that was held on Friday morning.

See Figure 4 for a summary of the progress.

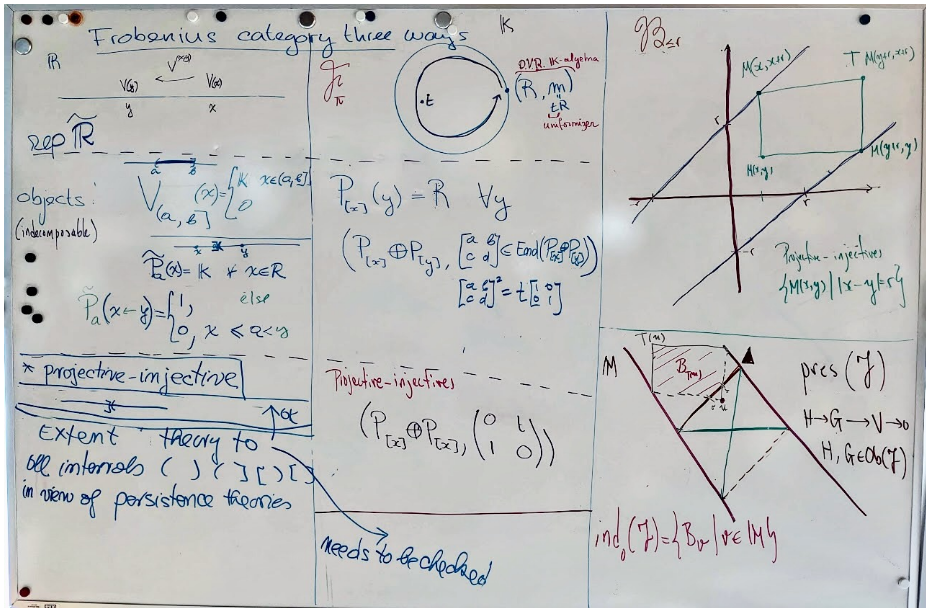

Figure 4: Poster summarizing progress of group 4 (Section 4.4).

4.5 Möbius inversion and persistence

Goals. There are several different proposals for using the Möbius inversion to generalize the classical persistence diagram. What are the pros and cons of each? Study algorithms for computing these various notions of generalized persistence diagrams. Study the stability of generalized persistence diagrams. Are there other applications of Möbius inversions to TDA or applied topology? For example, merge trees, dendrograms, Reeb graphs, etc. Möbius homology categorifies the Möbius inversions and it can be defined using projective/injective resolutions. There are now fast algorithms for computing minimal resolutions. Can we bring these ideas together?

After interesting discussions in the beginning of the workshop, participants from this group ended up joining other groups, as these topics had connections to various other ones.

4.6 Pairings and couplings between signed bars, relative Betti tables, or curves

Initial goals. Study the structure and possible pairing between segments of birth and death curves (for generalized diagrams of 2-filtrations). More generally, are there natural pairings/couplings between multigraded Betti numbers, relative Betti numbers, or birth and death curves? What are the implications for stability results for signed descriptors? What are the implications for the interpretability of signed descriptors?

Progress. We discussed the structure of the Möbius inversion/signed barcode and specifically the existence of birth and death curves which organize the intervals in the support of those functions. We discussed the different partitions of the birth/death curves into segments that emerge during the computation and how they relate to the relative Betti numbers in the projective resolution of the persistence module.

See Figure 5 for a summary of the progress.

4.7 Representation theory in persistence

Goals.

- What should finite-tame-wild mean for infinite posets and non-algebraically closed fields?
- What should the Brauer–Thrall conjectures say for infinite posets and non-algebraically closed fields?

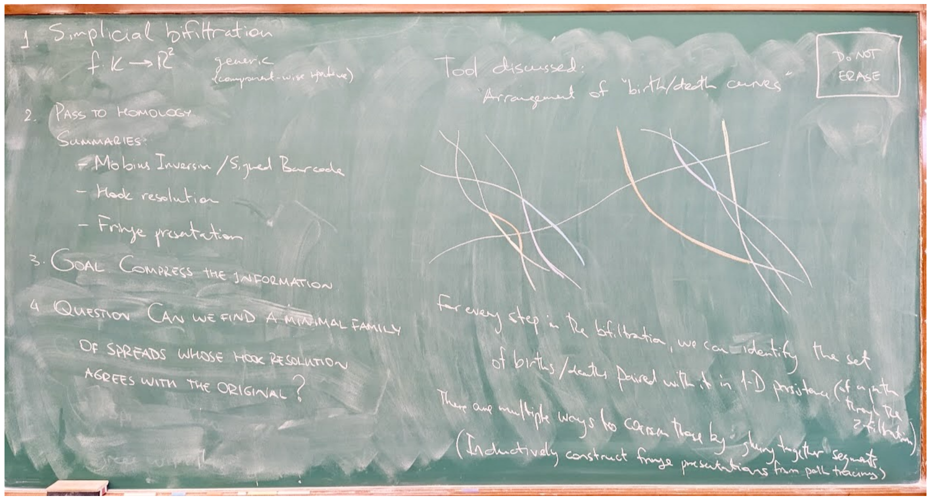

Figure 5: Poster summarizing progress of group 6 (Section 4.6).

- From invariants of persistence modules to invariants of algebras. Is there a way to leverage the invariants of persistence modules introduced in TDA to build new invariants of algebras that could be useful to the representation theory community?
- Interleavings in representation theory. What should interleavings be in the world of representations of algebras, and what uses may they have?

See Figure 6 for a summary of the progress.

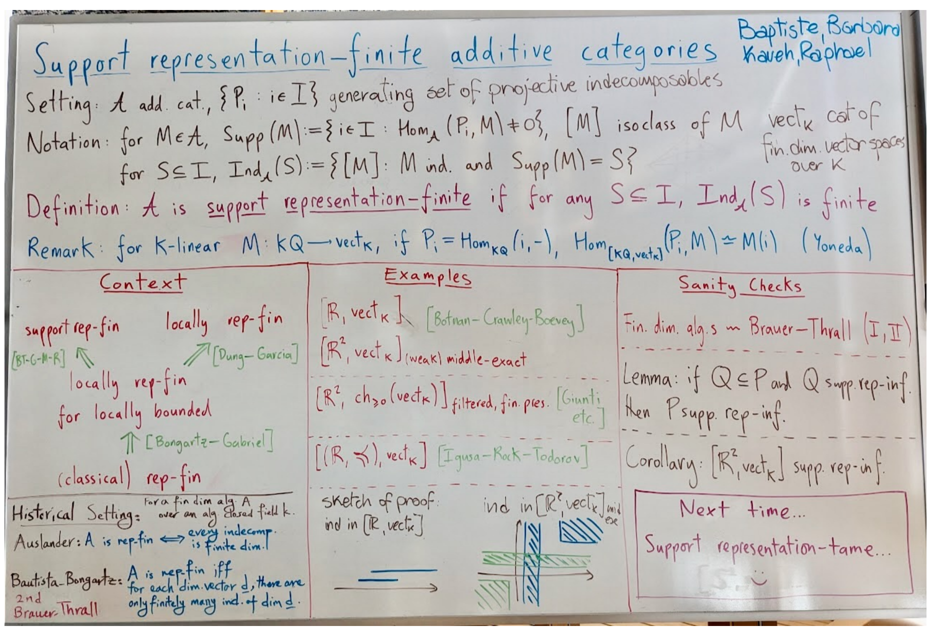

Figure 6: Poster summarizing progress of group 7 (Section 4.7).

5 Hybrid delivery format

The talks were accessible to all participants (on-site and online). One problem with hybrid format is that online participants come from various time zones, located for instance in Europe or Asia. We uploaded slides of the presentations so that online participants have access to the material in their preferred time, and managed to respond to questions coming online during day or night.

As for the working groups, we as organizers were humbled by the commitment and excitement that the working group leaders and each on-site participant demonstrated during the week at BIRS. We organized poster presentations at the last morning session, and observing the amount of energy each group created to design their poster and presenting their work was a very rewarding experience. The facilities and the setting at BIRS provide an ideal environment to enable such interaction and synergy among a small group of researchers who dedicate the whole week to exchange ideas and work together. We feel it would be very difficult to implement that in an online environment.

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