PIMS-BIRS TeamUp: Quantum State Transfer 24frg202

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1 Overview of the Field

The idea of evolving a quantum state through vertices of a decision tree via a continuous-time quantum walk was established two and a half decades ago [8]. A quantum walk is the quantum mechanical analogue of a classical random walk. More specifically, a continuous-time quantum walk on a graph X is described by a time-dependent transition matrix $U(t) = \exp(itH)$, where H is the Hamiltonian describing the quantum system, typically taken to be the adjacency matrix, Laplacian matrix, or signless Laplacian associated to the graph. While the work of [8] is focused on quantum algorithms, transport problems in quantum spin networks were considered in [3]. In this setting, the vertices of the graph are spins and the edges of the graph connect spins which interact. The ability to transfer a quantum state reliably from one location to another (within, say, a quantum computer), as well as generating entangled states, are important tasks to achieve in quantum spin systems. Both the situations described by [8] and [3] amount to the same thing—the study of fidelity of quantum state transfer, where the probability of state transfer between vertices a and b at time t is $p_{a,b}(t) = |\mathbf{e}_a^T \exp(itH)\mathbf{e}_b|^2 = |(U(t))_{a,b}|^2$, where $\mathbf{e}_a$ is the vector with 1 in the ath component and zeros elsewhere, and similarly for $\mathbf{e}_b$. The probability of state transfer is a number between 0 and 1 that measures the closeness of two quantum states, which determines the accuracy of the quantum state transfer. Much work has been done on the subject of quantum state transfer since these seminal works, developing perfect state transfer [5, 6, 9, 13], pretty good state transfer [10, 11, 16, 7], fractional revival [1, 2], and variants.

2 Recent Developments and Open Problems

Until very recently, work done on the topic of quantum state transfer has considered a spin propagating through vertices in the graph. However, [4] considers dynamics between certain linear combinations of two states, specifically pair state transfer from $\mathbf{e}_a - \mathbf{e}_b$ to $\mathbf{e}_\alpha - \mathbf{e}_\beta$, and plus state transfer from $\mathbf{e}_a + \mathbf{e}_b$ to $\mathbf{e}_\alpha + \mathbf{e}_\beta$. Work done in [14] takes this a step further, exploring the notion of an *s-pair state* of the form $\mathbf{e}_a + s\mathbf{e}_b$, where $s \in \mathbb{C}$, developing the theory of perfect s-pair state transfer in continuous quantum walks.

From a physics point of view, the vector $\mathbf{e}_a$ represents excitation of spin a, and so the state of an n-spin system can be represented as $|1\rangle_a|0\rangle_b|0\rangle_c \cdots \in \mathbb{C}^{2^n}$. Similarly, $\mathbf{e}_b$ corresponds to $|0\rangle_a|1\rangle_b|0\rangle_c \cdots$, where we have, without loss of generality, reordered our vertices for convenience so that the vertices a and

b appear first; indeed, vertices a and b need not be adjacent, and it is desirable to have them physically far apart in the network, so that one can transfer the quantum state of interest a far distance. The state $\frac{1}{\sqrt{1+s^2}}(|1\rangle_a|0\rangle_b + s|0\rangle_a|1\rangle_b)$ is a pair of entangled qubits (assuming $s \neq 0$) in the 1-excitation subspace $\mathbb{C}^n$, represented by the corresponding s-pair state $\frac{1}{\sqrt{1+s^2}}(\mathbf{e}_a + s\mathbf{e}_b)$. In what follows, we drop the normalization factor for ease of discussion, as it does not play a significant role.

Vertices a and b of a graph X are strongly cospectral (with respect to a matrix M associated to X) if $E_j\mathbf{e}_a = \pm E_j\mathbf{e}_b$ for each j, where the E_j's are orthogonal projection matrices onto each distinct eigenspace, and M is often taken to be the adjacency, Laplacian, or signless Laplacian of the graph (though other symmetric matrices may be used). Identifying strong cospectrality between vertices is an important tool in the study of quantum state transfer [12]. Furthermore, work done in [14, Theorem 2.3] shows that strong cospectrality of s-pair states is a necessary condition for perfect s-pair state transfer; their work also uncovered infinite families of graphs exibiting s-pair state strong cospectrality, periodicity, and perfect state transfer.

Because the study of s-pair states is very new, there are many open problems in the area. It is of interest to identify what results in the literature extend (or don't extend) from vertices to edges. Given the importance of strong cospectrality, one can explore analogous results on strongly cospectral s-pair states.

Work of [15] considers pair state transfer relative to the adjacency matrix of a graph, finding infinitely many trees with perfect pair state transfer. Further examples of families of graph having strong copsectrality of s-pair states (or not having strong copsectrality of s-pair states) would be a useful first step in identifying which graphs have/do not have perfect s-pair state transfer. One can explore s-pair state transfer for various Hamiltonians—the adjacency matrix, the Laplacian, the signless Laplacian, and so on.

3 Scientific Progress Made

Given that the numerics for small graphs show that perfect pair state transfer occurs more often than perfect vertex state transfer [4], the study of pair state transfer, and more generally, s-pair state transfer merits further investigation. The two-week Teamup mainly focused on finding examples of s-pair-state transfer, with the goal of providing infinite families of graphs allowing for quantum state transfer between these more general states. Specifically, it was noted that, although pair and plus states have been considered, quantum state transfer between a plus state and a pair state has not been explored; consequently, the group primarily focused on state transfer properties between s-pair states and $-s$-pair states. However, state transfer between states of the type $\mathbf{e}_a + s\mathbf{e}_b$ and $\mathbf{e}_\alpha + r\mathbf{e}_\beta$ (for vertices a, b, α, β and scalars r, s) was also considered, and the analogous definition of strong cospectrality — for each eigenprojection matrix E_j, $\exists\gamma_j$ with $|\gamma_j| = 1$ such that $E_j(\mathbf{e}_a + s\mathbf{e}_b) = \gamma_j E_j(\mathbf{e}_\alpha + r\mathbf{e}_\beta)$ — was investigated, leading to generalisations of results from the case of real r, s to complex r, s. As it turns out, in both the real and the complex settings, strong cospectrality (a necessary condition for perfect quantum state transfer) only holds provided $|r| = |s|$. However, some major differences arise in the complex setting, most notably, monogamy of perfect state transfer is lost. We also provide a number of infinite families of graphs exhibiting perfect state transfer between an s-pair-state transfer and a $-s$-pair-state in each of the $s \in \{\mathbb{R}, \mathbb{C}\}$ settings.

4 Outcome of the Meeting

The work done during the Teamup further developed quantum pair state transfer and underscores the utility of this type of transfer. The group will continue to meet regularly by way of video conferencing, with the goal of writing a manuscript for peer review.

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