PIMS-BIRS TeamUp: Twisted Tensors of Hopf Algebras (24rit600)

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1 Introduction

Representation theory studies the symmetries arising from algebraic objects. It began with the study of the actions of symmetric groups on regular polygons, and its modern treatment encompasses automorphic forms, invariant theory, harmonic analysis, as well as applications to mathematical physics. In this program we delved into quantum symmetries, namely the representation theory of Hopf algebras, and its ramifications.

The purpose of this program was to study the deformations arising from tensor products of Hopf algebras, and understanding their relation with the deformations arising from tensor products of Frobenius algebras, at a categorical level. These algebras have had a major impact in the modern approach to quantum symmetries, arguably the most imortant one being their uses to construct topological invariants. This first appeared in work of Majid [5], where he employed quantum groups to construct 3-dimensional topological quantum field theories. The modern incarnation of these applications is exemplified in Turaev–Virelizier [8], where they use spherical fusion categories instead. The symmetric monoidal equivalence between commutative Frobenius algebras and 2-dimensional topological quantum field theories is well known by Abrams [1] and Ocal [6], but the topological meaning of noncommutative Frobenius algebras remains elusive. A natural way of introducing this noncommutativity in a controlled way is to consider the deformations of tensor products of algebras proposed by Čap–Schichl–Vanžura [3], known as twisted tensor products, as well as their dual, which deserve the name cotwisted tensor products.

Understanding twisted and cotwisted tensor products of Frobenius algebras was pioneered by Ocal–Oswald [7], where they characterized certain classes of them. Their techniques are valid at the level of algebra and coalgebra objects in braided monoidal categories, establishing a path to work with the aforementioned deformations at a categorical level. This program brought together these authors to build upon their previous efforts.

2 Algebraic objects in monoidal categories

The basic insight of representation theory is that, given an algebra A over a field k, one can understand it through its *modules*. These are vector spaces M over the same field k equipped with a k linear map $\rho_M : A \otimes M \to M$ compatible with the multiplication m and the unit u of A. Namely, the equalities $(\rho_M)(\rho_M \otimes \mathrm{id}_A) = (\rho_M)(\mathrm{id}_M \otimes m)$ and $(\rho_M)(\mathrm{id}_M \otimes u) = (\mathrm{id}_M)(\mathrm{id}_M \otimes \rho_k)$ hold. As the above presentation suggests, all these notions can be given in terms of objects and morphisms in the category of k vector spaces Vec_k, as well as the commutativity of certain diagrams (or equivalently, the equality of certain morphisms in

Vec_k). This hints at the concepts of algebra and module being applicable within a larger family of categories admitting a generalization of the tensor product.

These are known as *monoidal categories*. They are a category C equipped with a bifunctor $\otimes : \mathsf{C}\times\mathsf{C} \to \mathsf{C}$ called the *tensor product*, a collection of natural isomorphisms $\alpha_{X,Y,C} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ for each triple of objects X, Y, Z in C called the *associativity morphisms*, a distinguished object $\mathbf{1}$ of C called the *unit*, a collection of natural isomorphisms $l_X : \mathbf{1} \otimes X \to X$ called the *left unitors*, and a collection of natural isomorphisms $r_X : X \otimes \mathbf{1} \to X$ called the *right unitors*, satisfying the pentagon and the triangle axioms.

$$(\mathrm{id}_W \otimes \alpha_{X,Y,Z})(\alpha_{W,X\otimes Y,Z})(\alpha_{W,X,Y} \otimes \mathrm{id}_Z) = (\alpha_{W,X,Y\otimes Z})(\alpha_{W\otimes X,Y,Z}) \qquad \text{(Pentagon axiom)}$$

$$r_X \otimes \mathrm{id}_Y = (\mathrm{id}_X \otimes l_Y)(\alpha_{X,\mathbf{1},Y}) \qquad \text{(Triangle axiom)}$$

For simplicity, in the rest of this report we will only consider *symmetric* monoidal categories. These are monoidal categories C equipped with a collection of natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ for each pair of objects X, Y in C called the *symmetry morphisms*, satisfying the left hexagon, right hexagon, and symmetry axioms.

$$(\alpha_{Y,Z,X})(c_{X,Y\otimes Z})(\alpha_{X,Y,Z}) = (\mathrm{id}_Y \otimes c_{X,Z})(\alpha_{Y,X,Z})(c_{X,Y} \otimes \mathrm{id}_Z) \qquad \text{(Left hexagon axiom)}$$

$$(\alpha^{-1}_{Z,X,Y})(c_{X\otimes Y,Z})(\alpha^{-1}_{X,Y,Z}) = (c_{X,Z} \otimes \mathrm{id}_Y)(\alpha^{-1}_{X,Z,Y})(\mathrm{id}_Z \otimes c_{X,Y}) \qquad \text{(Right hexagon axiom)}$$

$$(c_{X,Y})(c_{Y,X}) = \mathrm{id}_{X\otimes Y} \qquad \text{(Symmetry axiom)}$$

Fixing a symmetric monoidal category C for the rest of this report, we can now consider all the objects of interest for representation theory (and quantum symmetries in particular) within it.

An *algebra object* in C is an object A of C equipped with a morphism $m : A \otimes A \to A$ in C called *multiplication* and a morphism $u : \mathbf{1} \to A$ in C called *unit* satisfying the associativity, left unit, and right unit axioms.

$$(m \otimes \mathrm{id}_A)(m) = (m)(\mathrm{id}_A \otimes m)(\alpha_{A,A,A}) \qquad \text{(Associativity axiom)}$$

$$(m)(u \otimes \mathrm{id}_A) = (\mathrm{id}_A)(l_A) \qquad \text{(Left unit axiom)}$$

$$(m)(\mathrm{id}_A \otimes u) = (\mathrm{id}_A)(r_A) \qquad \text{(Right unit axiom)}$$

Its dual notion is called a *coalgebra object* in C. Namely, it is an algebra object in the opposite category C^{op}, which has a *comultiplication* Δ and a *counit* ϵ satisfying the coassociativity, left counit, and right counit axioms. We are interested in the following ways of making the structures of an algebra object A in C and a coalgebra object A in C compatible.

A *Frobenius algebra* in C is an object A of C equipped with a multiplication m, a unit u, a comultiplication Δ, and a counit ϵ, satisfying that A is an algebra object in C, A is a coalgebra object in C, and the Frobenius axiom.

$$(\mathrm{id}_A \otimes m)(\Delta \otimes \mathrm{id}_A) = (\Delta)(m) = (m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta) \qquad \text{(Frobenius axiom)}$$

This condition is, in fact, a topological restriction. Interpreting the multiplication as a (thinned) cup and the comultiplication as a (thinned) cap, it is equivalent to the fact that the string diagram obtained by stacking can be modified up to isotopy without altering the final result.

A *bialgebra* in C is an object A of C equipped with a multiplication m, a unit u, a comultiplication Δ, and a counit ϵ, satisfying that A is an algebra object in C, A is a coalgebra object in C, and the bialgebra axioms.

$$(m \otimes m)(\mathrm{id}_A \otimes c_{A,A} \otimes \mathrm{id}_A)(\Delta \otimes \Delta) = (m)(\Delta) \qquad \text{(Bialgebra axiom for } m \text{ and } \Delta)$$

$$(\iota)(\epsilon \otimes \epsilon) = (\epsilon)(\Delta) \qquad \text{(Bialgebra axiom for } m \text{ and } \epsilon)$$

$$(\Delta)(u) = (u \otimes u)(\iota) \qquad \text{(Bialgebra axiom for } u \text{ and } \Delta)$$

$$\mathrm{id}_{\mathbf{1}} = (\epsilon)(u) \qquad \text{(Bialgebra axiom for } u \text{ and } \epsilon)$$

An important observation is that $l_{\mathbf{1}} =: \iota := r_{\mathbf{1}}$ in C, so there is only one canonical isomorphism $\iota : \mathbf{1}\otimes\mathbf{1} \to \mathbf{1}$. Bialgebras are fundamental objects because their categories of modules classify monoidal categories.

A *Hopf algebra* in C is a bialgebra A in C equipped with an invertible morphism $S : A \to A$ satisfying the antipode axiom.

$$(m)(\mathrm{id}_A \otimes S)(\Delta) = (u)(\epsilon) = (m)(S \otimes \mathrm{id}_A)(\Delta) \qquad \text{(Antipode axiom)}$$

Hopf algebras are fundamental objects because their categories of modules classify finite tensor categories admitting a fiber functor. In fact, the categories of modules of weak Hopf algebras classify multifusion categories. The study of these more general objects over $\mathsf{C} = \mathsf{Vec}_k$ is a rich and fruitful modern approach to quantum symmetries, but little is known about them over a general symmetric monoidal category C.

A vital relation between Frobenius algebras and Hopf algebras is that when $\mathsf{C} = \mathsf{Vec}_k$ every Hopf algebra is a Frobenius algebra, but with a different comultiplication and counit (the multiplication and unit remain unchanged). The only exception is the object $\mathbf{1}$ equipped with $m = \iota$, $u = \mathrm{id}_{\mathbf{1}}$, $\Delta = \iota^{-1}$, $\epsilon = \mathrm{id}_{\mathbf{1}}$, and $\epsilon = \mathrm{id}_{\mathbf{1}}$, which is simultaneously a Fobenius algebra and a Hopf algebra.

3 Scientific progress made

The key contribution of this program is the study of twisted and cotwisted tensor products of Hopf algebras and Frobenius algebras within monoidal categories.

Given A and B algebras in C, an invertible morphism $\tau : B \otimes A \to A \otimes B$ in C is said to be a *twist* when it is compatible with the multiplications m_A and m_B and the units u_A and u_B of A and B respectively. Namely, the equalities $(\tau)(m_B \otimes m_A) = (m_A \otimes m_B)(\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B)(\tau \otimes \tau)(\mathrm{id}_B \otimes \tau \otimes \mathrm{id}_A)$, $(u_A \otimes \mathrm{id}_B)(c_{B,\mathbf{1}}) = (\tau)(\mathrm{id}_B \otimes u_A)$, and $(\mathrm{id}_A \otimes u_B)(c_{\mathbf{1},A}) = (\tau)(u_B \otimes \mathrm{id}_B)$ hold. When τ is a twist, the object $A \otimes B$ in C equipped with $m = (m_A \otimes m_B)(\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B)$ and $u = (u_A \otimes u_B)(\iota)$ is an algebra object in C, called the *twisted tensor product* of A and B.

The dual notions to twist and twisted tensor product are *cotwist* and *cotwisted tensor product*. They arise by taking A and B algebras in C^{op}, taking θ a twist in C^{op} (which is a morphism $\theta : A \otimes B \to B \otimes A$ in C), and taking the resulting twisted tensor product of A and B in C^{op}. In this categorical framework, the fact that $A \otimes B$ equipped with $\Delta = (1 \otimes \theta \otimes 1)(\Delta_A \otimes \Delta_B)$ and $\epsilon = (\iota)(\epsilon_A \otimes \epsilon_B)$ is a coalgebra object in C is an immediate corollary of the aforementioned result for twisted tensor products. This ability to reason simultaneously for algebras and coalgebras is one of the desirable advantages we seek by working categorically.

The twisted tensor product of A and B via τ is denoted by $A \otimes_\tau B$, and the cotwisted tensor product of A and B via θ is denoted by $A \otimes^\theta B$. The case when $\mathsf{C} = \mathsf{Vec}_k$ is the context in which twisted and cotwisted tensor products were originally conceived, and it has recently regained attention in the literature. Among others, there have been attempts at classifying them, at computing their invariants, and at determining whether they inherit any meaningful properties. Our interests during this program align with these later ideas.

Understanding any facet of twisted or cotwisted tensor products in full generality is usually a near impossible task. They are vast families of algebras encompassing triangular algebras, smash products of Hopf algebras, and many of the known biproduct constructions, which makes finding a common behavior challenging. The fact that our categorical approach has been successful at these tasks for both *twisted and cotwisted tensor products* $A \otimes_\tau^\theta B$ attests to its power. In this program we concluded the characterization of when twisted and cotwisted tensor products inherit the structure of a Frobenius algebra, and we provided novel conditions under which the twisted and cotwisted tensor product of Hopf algebras inherite the structure of a Hopf algebra. These can be summarized in the following results.

Theorem 1 *Let $\mathcal{C}$ be a monoidal category, let A and B be Frobenius algebras in $\mathcal{C}$, let $\tau : B \otimes A \to A \otimes B$ be a twist in $\mathcal{C}$, and let $\theta : A \otimes B \to B \otimes A$ be a cotwist in $\mathcal{C}$. Then:*

$A \otimes_\tau^\theta B$ is a Frobenius algebra in $\mathcal{C}$ if and only if $\theta = \tau^{-1}$.

Theorem 2 *Let $\mathcal{C}$ be a braided monoidal category, let A and B be Hopf algebras in $\mathcal{C}$, let $\tau : B \otimes A \to A \otimes B$ be a twist in $\mathcal{C}$, and let $\theta : A \otimes B \to B \otimes A$ be a cotwist in $\mathcal{C}$. Suppose that either*

$$(\theta \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes c_{B,B})(\tau \otimes \mathrm{id}_B) = (\mathrm{id}_B \otimes \tau)(c_{B,B} \otimes \mathrm{id}_A)(\mathrm{id}_B \otimes \theta), \text{ or}$$
$$(\theta^{-1} \otimes \mathrm{id}_A)(\mathrm{id}_B \otimes c_{A,A})(\tau^{-1} \otimes \mathrm{id}_A) = (\mathrm{id}_A \otimes \tau^{-1})(c_{A,A} \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \theta^{-1}).$$

Then:

$A \otimes_{\tau}^{\theta} B$ is a Hopf algebra.

In particular, when $\tau = c_{B,A}$ or $\theta = c_{A,B}$ then a twisted and cotwisted tensor product of Hopf algebras is always a Hopf algebra. This is a generalization of the well known result that a tensor product of Hopf algebras can be endowed with a Hopf algebra structure by swapping the two algebras when multiplying and comultiplying. This particular case of Theorem 2 already recovers useful examples, such as the group algebra of the semidirect product of groups, the Drinfeld double of a Hopf algebra, and many other commutative and cocommutative Hopf algebras.

4 Outcomes and conclusions

This program achieved its goal of characterizing when the twisted and cotwisted tensor product of Frobenius algebras inherits a Frobenius algebra structure from its components, and also achieved its goal of providing new sufficient conditions under which the twisted and cotwisted tensor product of Hopf algebras inherits a Hopf algebra structure from its components.

A curious observation is that the conditions of Theorem 2 are topological in nature. Using the string diagrams native to monoidal categories, the maps τ and θ can be interpreted as special invertible crossings, and the conditions become generalized third Reidemeister moves. Since (as we hinted above) Hopf algebras are fundamentally not topological in nature, this adds a layer of nuance that deserves further exploration. In particular, this may point at deeper relations with other topological conditions that have also played a role in the literature of both monoidal categories and quantum groups, such as the Yang–Baxter equation.

5 A personal note from the organizers

The organizers would like to wholeheartedly thank the BIRS directorship and staff for the opportunity of having this program. It was a deeply rewarding experience that we enjoyed to the fullest. The work we conducted during our stay is now at the core of our research programs.

6 List of participants

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