Cartan subalgebras in operator algebras, and topological full groups

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1 Overview

The focus of this workshop was the role of Cartan subalgebras in two branches of operator algebras—C^*-algebras and von Neumann algebras—together with links between these and geometric group theory. The study of existence and uniqueness of Cartan subalgebras, and what existence and uniqueness can tell us about the structure of the algebras, are very active fields of research, with deep connections to ergodic theory, topological dynamics and geometric group theory.

The link between Cartan subalgebras of operator algebras and geometric group theory is via generalized dynamical systems called *groupoids*, and their *topological full groups*. Recent breakthroughs show how they are related. Each Cartan subalgebra determines a groupoid and each groupoid has a topological full group. These topological full groups have provided examples that answered long-standing questions in group theory. Further, completing the circle, the homology theory of a topological full group provides K-theoretic information about the original operator algebra.

The workshop built on these exciting advances and was able to foster interactions between group theorists and operator algebraists, and between researchers in C^*-algebras and in von Neumann algebras–two areas of operator algebras that have recently reengaged with stunning success in the classification program for C^*-algebras.

2 Scientific Background and Recent Developments

A *Cartan pair* (A, B) is a C^*-algebra A with a very special maximal abelian subalgebra B. There are two classical examples: (1) the $n \times n$ matrices with the diagonal matrices and (2) the crossed product arising from a group acting effectively on a compact space X with the continuous functions on X. It turns out that all Cartan pairs come from a groupoid: for von Neumann algebras this is a Borel equivalence relation [10] and for C^*-algebras it is an étale groupoid [21]. The earlier von Neumann theory inspired the C^*-algebraic formulation. A breakthrough result of Matsumoto and Matui in 2014 used [21] to recover an important invariant called *flow equivalence* from a Cartan pair [14]. Since then there have been many papers on how to recover different types of dynamical invariants from Cartan pairs (for example, [4, 5, 6]).

Barlak and Li proved in 2017 that every nuclear C^*-algebra with a Cartan subalgebra belongs to the UCT class (a class of C^*-algebras satisfying a universal coefficient theorem in KK-theory) [1]. Since the question of whether every nuclear C^*-algebra satisfies the UCT is the one remaining hurdle in the classification program for simple C^*-algebras, this result is a major step forward and has generated substantial interest. Consequently, establishing existence of Cartan subalgebras is currently a hot topic. More generally, classifying the Cartan subalgebras with a given spectrum in a given C^*-algebra is an interesting, but usually hard, problem [2]. The classification of Cartan subalgebras of von Neumann algebras is more advanced: showing existence and/or uniqueness of a Cartan subalgebra is also vitally important [3, 19], but there is an additional focus on finding classes which do not have Cartan subalgebras, going back to the first such examples in [22].

In 2004, Nekrashevych discovered that the famous Thompson's group V embeds in the unitary group of the infamous Cuntz algebra $\mathcal{O}_2$ [17]. By showing that these unitaries normalize the Cartan subalgebra of $\mathcal{O}_2$, Matui showed that this group is the topological full group of the associated groupoid [16]. Because $\mathcal{O}_2$ is a prototype for many interesting classes of C^*-algebras [7, 12, 17], this has led to many more examples of interest to geometric group theorists [13, 18]. Going back to the classical example of a Cartan pair, Nekrashevych showed that the infinite dihedral group acting on the Cantor set gives a groupoid whose topological full group is the first example of a finitely generated infinite simple group of intermediate growth [18]. Further, Juschenko and Monod showed that the topological full group of a minimal action of $\mathbf{Z}$ on the Cantor space is the first example of a finitely generated infinite simple amenable group [11].

Matui has shown that the topological full groups of a large class of minimal étale groupoids remember the whole groupoids [16] and hence their C^*-algebras. There is a tight connection between the K-theory of the C^*-algebras of a groupoid, the homology of the groupoid itself, and the homology of the topological full group of the groupoid. Explicit links to the generalizations of the famous Thompson's groups have appeared in [13, 15], for example.

3 Open Problems

Some subproblems that were discussed and raised during the workshop include the following:

1. Two Cartan pairs are isomorphic if and only if their Weyl groupoids coincide. Are there some coarser but useful and computable invariants of Cartain pairs, for example, in KK-theory?

2. Can we describe all the Cartan subalgebras with a given spectrum in a given C^*-algebra? For example, in subhomogeneous or approximately finite-dimensional C^*-algebras? Results and techniques from von Neumann algebras could inform the strategy.

3. Can we compute the topological full groups of selected groupoids; for example, AF-groupoids, or groupoids associated to semigroups such as 1-vertex higher-rank graphs? What can these computations contribute to questions in geometric group theory?

4. Can we obtain novel purely group-theoretic invariants of topological full groups from invariants of C^*-algebras? For example, is there a purely group-theoretic characterization of the invariant that distinguishes the topological full groups associated to $\mathcal{O}_2$ and its splice $\mathcal{O}_{2,-}$?

5. The unitaries in a C^*-subalgebra that normalize a Cartan subalgebra may encode the topological full group acting on the Cartan subalgebra. What is the crossed product? An intermediate subalgebra?

6. There are classes of von Neumann algebras with no Cartan subalgebras. Are there analogous C^*-algebras? If so, what are the consequences for classification?

7. Is k-graph cohomology an invariant of the k-graph groupoid?

8. Which classifiable C^*-algebras admit a 0-dimensional Cartan subalgebra, a Cartan subalgebra for which the twist is trivial, a (0-dimensional or not) C^*-diagonal (with or without trivial twist)?

9. Is there an explicit demonstration of quasidiagonality of topological full groups of minimal $\mathbb{Z}$-actions?

10. Is there a relationship between coarse geometry and Cartan pairs for groupoids?

11. Find more examples of self-similar actions on graphs of rank 2 or more; how can one tell that they are new?
12. Can we make sense of self-similar actions of (discrete) groupoids on trees/forests? Find natural examples.
13. Can we make sense of self-similar partial-actions of groups on higher-rank graphs and their representations?
14. Can techniques arising from the study of Cartan subalgebras be used to study non-regular inclusions of C^*-algebras?
15. Do contracting self-similar actions yield subhomogeneous (nonabelian) Cartan subalgebras?
16. Is there a direct description of the topological full group of the universal groupoid of an inverse semigroup S in terms of S? How do properties of S correspond to properties of the group?
17. Find a group-theoretic invariant that witnesses the non-isomorphism of the topological full groups of the standard groupoids for $\mathcal{O}_2$ and $\mathcal{O}_{2,-}$. Find a K-theoretic invariant of Cartan pairs that distinguishes $\mathcal{O}_2$ with its standard Cartan from and $\mathcal{O}_{2,-}$ with its standard Cartan.
18. Find a "good" notion of the topological full group of a non-ample groupoid that connects appropriately with the C^*-algebra/normaliser picture.
19. Suppose (A,B) is a regular inclusion and J is an ideal of A having trivial intersection with B. If (A/J, B) is Cartan, must A contain a Cartan MASA?
20. Let G be a group and consider the C^*-algebra of partial representations of this group. This algebra can be described as a groupoid C^*-algebra (indirectly follows from [9]). Does the full group of this groupoid give any interesting information about G?
21. Suppose G is an étale groupoid with unit space X. What is the maximal subgroupoid of G whose reduced C^*-algebra contains $C_0(X)$ as a Cartan subalgebra? More generally, what is the largest subalgebra B of $C_r^*(G)$ that contains $C_0(X)$ as a Cartan subalgebra?

4 Scheduling, Speakers, and Presentation Highlights

The workshop was focused on creating new research collaborations rather than only reporting on existing work. We started on Monday with three introductory keynote talks, followed by a session to discuss open problems. Aside from a fourth keynote talk on Tuesday and 17 shorter talks of 20 minutes each, the workshop's schedule allowed for the participants to work in groups on any of the open problems that had been identified, or other related collaborations.

A survey that was sent to out by the organizers after the workshop shows that the participants very much appreciated the schedule: keynotes and problem sessions to set the scene; a few, short but high-quality talks to learn new advancements; and many hours to both work on existing projects and organically form new collaborations. More details about the survey are given in Section 7 below. Further appreciated was the diversity in the speaker and participant list, in terms of career stage, mathematical background, geographical location, and gender.

4.1 Keynote talks (50 minutes)

David Pitts on "Generalisations of Cartan subalgebras"

Renault's theorem (extended by Raad) says that a subalgebra D of a C^*-algebra A is a Cartan subalgebra precisely when there is a Hausdorff étale groupoid G and a twist Σ over G for which A is the reduced C^*-algebra of the twist, and D is the subalgebra consisting of functions supported on the unit space of G. However, ever since Connes' groundbreaking work on foliated manifolds and Paterson's work on C^*-algebras of inverse semigroups, and with increased intensity since the advent of self-similar actions and associated C^*-algebras, there has been intensifying interest in C^*-algebars associated to étale groupoids that are not Hausdorff but do have Hausdorff unit spaces. Professor Pitts described the rapidly-developing theory for characterising the pairs consisting of a C^*-algebra and an abelian subalgebra that arise in this situation, or in the more general situation where the twist Σ is replaced by a more general Fell bundle.

Dilian Yang on "Cartan subalgebras and topological full groups coming from higher-rank graphs"

The theory of C^*-algebras has historically been driven by the discovery of new classes of examples leading to the investigation of new behaviour. Similarly, ever since the landmark classification of finite simple groups, an enormous amount of work in geometric group theory has been invested in generating new classes of examples, particularly of simple infinite groups. One source of examples for both of these areas, which has already proven a rich vein, and which appears to have enormous unexplored potential, is the theory of topological full groups arising from étale groupoids. But this in turn begs the question of how to construct large classes of examples of such groupoids. One avenue is higher-rank graphs, the topological full groups of whose groupoids have already been demonstrated as capturing the Brin-Thompson groups and the integer Stein groups. Professor Yang discussed the fundamental theory of higher-rank graphs and the process whereby they give rise to groupoids and thence to topological full groups and Cartan subalgebras, and gave an overview of structure theory for these objects.

Sven Raum on "Cartan subalgebras from the von Neumann algebra point of view"

The recent success of, and interest in, Cartan pairs of C^*-algebas owes its origins, as so much of the theory of C^*-algebras does, to earlier work in von Neumann algebras. Specifically, Feldman and Moore characterised the abelian subalgebras of von Neumann algebras that arise as the algebra of Borel functions on the unit space of a Borel equivalence relation, providing a concrete way to describe coordinates for a given von Neumann algebra. It was this theorem that inspired Renault's work. Since the work of Feldman and Moore, the structure theory of Cartan subalgebras has been a key theme in von Neumann algebras, with particularly important work coming out of the question of existence, uniqueness, and classification of Cartan subalgebras in given von Neumann algebras. Professor Raum gave an overview of the main techniques, questions, and lines of investigation in this area, and described what the corresponding questions might look like for C^*-algebras. He also outlined what key points might make these questions more difficult in the context of C^*-algebras, and and what might be the major obstacles to adapting the existing von Neumann algebraic techniques to try to answer them.

Brita Nucinkis on "An introduction to Automorphism groups of Cantor algebras and their connection to topological full groups"

The idea of "convergent evolution" in biology describes the situation in which similarities between two genetically distinct species are explained by evolutionary responses to the same environmental forces. We sometimes see the same thing in mathematics, when two research areas, coming from completely different backgrounds, develop similar tools and techniques because, it turns out, they are both studying the same underlying structure, from different angles. Cartan pairs of C^*-algebras and geometric group theory provide an excellent example of this. A celebrated theorem of Matui, which in turn builds on Rubin's theorem for group actions, says roughly speaking that minimal effective Hausdorff ample groupoids and their topological full groups contain the same information. This perhaps explains why, in both areas, tools based on tree-like structures and their symmetries have emerged as means of constructing and analysing new examples. Professor Nucinkis gave an overview of the theory of Cantor algebras, which are constructed from such tree-like structures, how they can be used to answer questions and construct examples in geometric group theory, and how they relate to and generalise known constructions of groupoids and of C^*-algebras, including higher-rank graphs.

4.2 Research talks (20 minutes)

1. Alejandra Garrido on "Non-discrete topological full groups"

2. Adam Fuller on "Recovering elements from their supports in groupoid C^*-algebras"
3. Camila Sehnem on "A characterization of primality for reduced crossed products"
4. Xin Li on "Ample groupoids, topological full groups, algebraic K-theory spectra and infinite loop spaces"
5. Larissa Kroell on "The ideal intersection property for partial C^*-dynamical systems"
6. Dolapo Oyetunbi on "C^*-diagonals of Inductive Limits of 1-Dimensional Noncommutative CW-Complexes"
7. Matt Kennedy on "Intermediate subalgebras for reduced crossed products of discrete groups"
8. Karen Strung on "Classifiable C^*-algebras from dynamics"
9. Ilija Tolich on "Non-Hausdorff Groupoids and Purely Infinite C^*-algebras"
10. Diego Martinez on "Cartan subalgebras, cp maps and their inductive limits"
11. Jennifer Pi on "A speculative talk on model theory for Cartan inclusions"
12. Rufus Willett on "Cartan subalgebras in uniform Roe algebras"
13. Ying-Fen Lin on "Semi-Cartan subalgebras of twisted groupoid C^*-algebras"
14. Marcelo Laca on "Relative Topological Principality and the Ideal Intersection Property"
15. Ian Putnam on "Exotic Cartan subalgebras for familiar C^*-algebras"
16. Conchita Martinez-Perez on "Explicit presentations for some groups of automorphisms of Cantor algebras"
17. Becky Armstrong on "Representing topological full groups in Steinberg algebras and C^*-algebras"

5 Scientific Progress Made

The workshop was characterized by a lively exchange of ideas, with a lot of active discussions taking place not only during the presentations but also throughout the breaks and meal times. The open problem session on Monday provided all participants a good opportunity to propose a lot of open problems. This was especially fruitful after the keynote talks. The question box discussion session on Thursday afternoon contributed additional open problems. The daily working in group session offered the participants focused opportunities to delve into specific challenges.

6 Outcome of the Meeting

The workshop turned out to be very fruitful. There are about 16 new collaborations arising from the workshop, involving 27 participants (out of 40), and some of these collaborations are already reporting progress. A wide breadth of research topics were presented during the workshop, and based on feedback from the survey conducted after the workshop, a high percentage of participants said that they have learned a lot from the workshop. So overall, the goals of this workshop were achieved.

7 Feedback

Of the 40 workshop participants, 19 provided feedback to the survey sent out by the organizers after the workshop. Of the 19 respondents, 18 had earned a doctorate degree, and 15 were employed in academia. This workshop was a new experience for many of the participants. The following excerpts show that the feedback was very positive indeed.

7.1 Something liked

The overall theme of the feedback was that the workshop was amazing, had great diversity and balance; that they loved the allotted research time and welcoming atmosphere.

- "The workshop consisted of a perfect blend of research talks updating us on the current state-of-the-art as well as enough time to consider the open problems posed and make new connections with the other experts, as well as making initial attempts of solving these problems. From an information point-of-view I learnt a lot about the open problems being considered, and from the practical point-of-view I felt like I was given ample opportunity to attempt considering a subset of these problems as well as discussing the problems and possible approaches with other experts in the field. The workshop brought me closer to the people working in my area and I feel this has opened the door to continued collaboration and communication."
- "I thought the conference was very well organized and have no suggestions for the organizers themselves."
- "The workshop was very well scheduled and organised, with enough but not too many talks and plenty of time for discussions. The talks were interesting and really linked with the theme."
- "I had a marvelous (and productive) time, and I appreciate your efforts to connect folks from different backgrounds and to start collaborations."
- "The workshop had a good quality and quantity of talks. The keynote speakers were well chosen, and they did an excellent job in setting out the themes of the week. I think it was a very good idea to not over-fill the schedule with talks. There was ample time and energy for mathematical discussion. From what I saw, it looked like people took advantage of this."
- "I really enjoyed the atmosphere and that there was a lot of time for discussion. It was one of the first workshops I have been to where I had discussions with various participants on multiple occasions. I appreciate the mix of participants you brought together and the speakers you have picked. I also liked that the talks other than the keynotes were kept to 20 minutes with ample time for questions. This made talks outside my area easier to follow."
- "I liked the structure of the workshop, which included all the plenary talks earlier in the week so we could have an overall idea of the topics of interest. I also liked having time slotted for open problems and working in groups."
- "I enjoyed the collaborative and diverse atmosphere at the workshop and meeting new people in related areas. I appreciated that the aim of the talks was more introductory and big-picture than usual conference talks."
- "It was good that there was plenty of discussion time during the workshop. Also the active exchange of questions beyond the usual questions after a talk or during conversations in small groups was a positive aspect of the conference."
- "The place was of course amazing, but the atmosphere was the best. I think the participants were all eager to learn and share what they knew, and that's great. The fact that the topic was somewhat concrete also helped a ton."
- "The workshop was fabulous! The Banff Center is a wonderful place to do mathematicsits world class. I thought the organizers made excellent decisions regarding participant invitations and the talks were great. The fact that there was near gender balance amongst the participants was refreshing. I appreciated the time built in for discussions and the problem sessions."
- "I appreciated that the topic and speakers had been well curated and I felt that my perspective on this area has been significantly broadened. Even when I didn't necessarily follow the detail of a talk it provided context and motivation for other talks we had heard. And allowed me to better appreciate the area."

- "There was a nice mix of interrelated areas which means we were exposed to different viewpoints, questions and tools. As with other workshops of a similar flavour that I have attended (organised by similar people), the participants have been very welcoming and the atmosphere was one that fostered participation and collaboration between different areas. The discussion sessions and open problems were helpful to identify common interests or possible collaborations, sometimes with unexpected people."

- "The informal format was very enjoyable and gave ample time to either rest or discuss mathematics. I used these opportunities to actively ask several people for clarifications on their talks and to discuss related mathematical ideas. I also used some of the informal time to rest, which helped with paying attention to talks! Often in conferences, too many talks means (for me) an inability to actively engage with each topic. The talks were also of a high quality, and I appreciated that all organizers kept speakers running on time so that we didn't fall behind on schedule."

- "The friendly atmosphere and the emphasis on creating collaboration."

- "It was a notably constructive environment. Compared to other conferences I have been to, it seemed to be very successful in terms of encouraging people to talk to each other, and to work together. The choice of topic was quite good for that: focused enough that most people shared at least some background, but broad enough so that everyone was also seeing new things. The choice of participants may have also helped with the atmosphere. The organizers also did a good job of encouraging the speakers to give talks on topic: very often at a conference people talk about something barely connected from the conference topic, which can affect coherency.

  I thought there was a good mix between time for talks, and time for collaboration / informal interactions.

  It was one of the (and probably the) best conference I have been to in terms of diversity of the attendees. For example, there were a good number of junior participants, the geographic diversity was notable, and the number of female participants was exceptionally high for an operator algebras conference (at least, for the ones I normally go to)."

7.2 New collaborations

As mentioned in Section 6, the feedback indicated that there are about 16 new collaborations arising from the workshop involving 27 of the 40 participants.

7.3 Something learned

- "For someone whose primary research area is not operator algebras, I mainly learned things which will broaden by perspective and understanding of the wider context of my operator-algebra adjacent research, rather than tools I can directly use. In particular:

  I better understood why Cartan subalgebras are useful, and gained some understanding of the kinds of questions operator algebraists are interested in the area.

  I also learned that model theory can be useful to study things like the UTC problem which I had no idea about."

- "It is really interesting to learn Cartan subalgebras in von Neumann algebras and properties/results about the ideal intersection property."

- "I learnt a lot. I had little-to-no experience with fundamental groups before the meeting. The talks in the workshop led to David Pitts and I discussing something we would not have otherwise considered."

- "I have found a new appreciation for groupoids and semigroup actions, the latter fuelled by several conversations with participants."

- It was really interesting to see the perspective of people working in group theory."

- “I knew nothing about topological full groups before the conference, so I learned some about them. I also learned more about graph moves and k-graph moves.”
- “The talks, whilst generally in my area of knowledge, offered new approaches and ideas to certain structures that taught me a lot. Also the one on one discussions taught me numerous new concepts as well.”
- “More generally I felt that I got a much wider appreciation of the field which helped me appreciate the significance and context of papers I’d been reading.”
- “I learned about techniques for analysing von Neumann Cartan subalgebras.”
- “I have been pointed to many references on self-similar inverse monoids/groupoids that have been thought about in the C*-algebra community and that could be sources of new non-discrete totally disconnected locally compact full groups.”
- “I have also learned from conversations with C*-algebraists that there is still a need for examples coming from groups with certain properties, so I shall look into possible group constructions within the classes I am familiar with.”
- “I do not come from a C*-dynamics or group theory background, so I learned a lot of new topological group terminology. I was also exposed to a lot of new questions such as classifying intermediate subalgebras of C*-algebra inclusions, and what a good notion of the modular automorphism group for C*-algebras would be.”
- “I have tended to avoid twists and non-Hausdorff groupoids in the past, partly due to intimidation. I learnt quite a lot about these from both talks and informal discussions (particularly from Astrid an Huef about twists), and I also had very useful discussions on noncommutative convexity (with Matthew Kennedy), which is completely new to me, and seems quite powerful for problems I am interested in.”
- “I learned *a lot* about current research into Cartan subalgebras. Very useful if a little daunting.”

8 Format

This was an in-person workshop only, with no online component.

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