Probabilistic methods in hyperbolic geometry

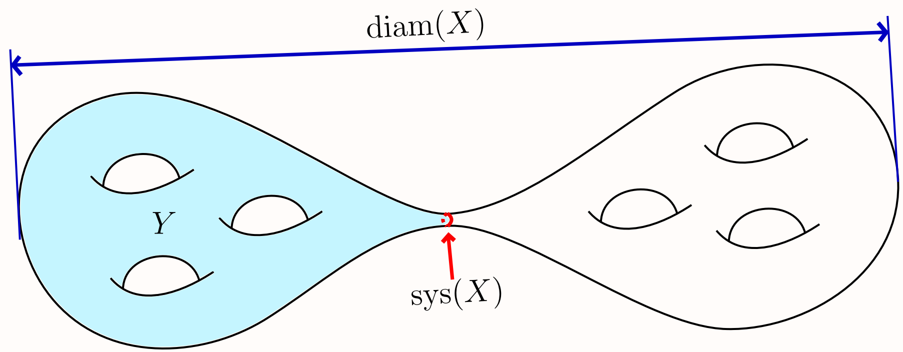
Joint with Thomas Budzinski & Nicolas Curien, Maxime Fortier Bourque, Mingkun Liu

Bram Petri

Knot Theory Informed by Random Models and Experimental Data

April 2, 2024

Extremal problems:



Definition: X a closed hyperbolic d-manifold.

- The systole: the length of the shortest closed geodesic and the kissing number: the number of geodesics realizing it,
- the diameter:

$$\mathrm{diam}(X) = \max\{d(x,y);\ x,y \in X\},$$

- the Cheeger constant (or isoperimetric constant):

$$h(X) = \inf \left\{ \frac{\mathrm{vol}_{d-1}(\partial Y)}{\mathrm{vol}_d(Y)};\ \begin{array}{c} Y \subset X \text{ submanifold} \\ \mathrm{vol}(Y) \leq \mathrm{vol}(X)/2 \end{array} \right\},$$

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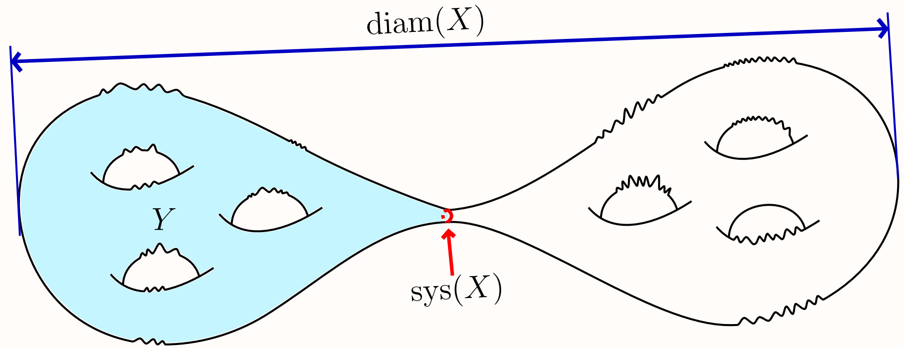

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- the spectral gap: the smallest non-zero eigenvalue of $\Delta = -\mathrm{div} \circ \mathrm{grad} : C^\infty(X) \to C^\infty(X)$ and its multiplicity.

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Fix the dimension $d \geq 2$.

Hyperbolic packing and kissing number problems:

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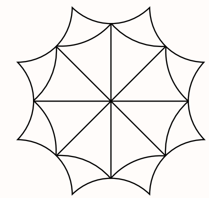
Spectral problems:

$$\sup\{\lambda_1(X);\ \mathrm{vol}(X) \geq v\} \quad \text{and} \quad \max\{m_1(X);\ \mathrm{vol}(X) \leq v\} \quad ?$$

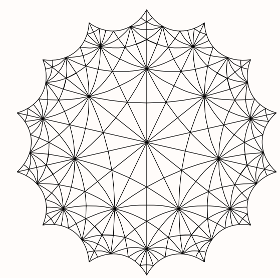
Lots of previous work: Huber '74, Cheng '75, Huber '76, Buser '77, Huber '80, Yang–Yau '80, Jenni '84, Burger Colbois '85, Brooks '88, Burger–Buser–Dodziuk '88, Colbois–Colin-de-Verdière '88, Burger '90, Schmutz '93, Schmutz '94, Buser–Sarnak '94, Bavard '96, Bavard '97, Schmutz-Schaller '97, Adams '98, Hamendstädt '01, Hamenstädt–Koch '02, Kim–Sarnak '03, Casamayou-Boucau '05, Katz–Schaps–Vishne '07, Otal '08, Gendulphe '09, Otal–Rosas '09, Parlier '13, Strohmaier–Uski '13, Fanoni–Parlier '15, Gendulphe '15, Cook '18, Petri–Walker '18, Petri '18, Hide–Magee '21, Jammes '21, Bonifacio '21, Kravchuk–Mazac–Pal '21, Wu–Xue '21, Lipnowski–Wright '21, Fortier Bourque–Rafi '22, Magee–Naud–Puder '22, Anantharaman–Monk '23, and many others.

Known maximizers:

	Systole	**Kissing number**	λ_1	$\mathbf{m_1}$
genus 2	Bolza surface **[Jenni '84]**	Bolza surface 24, **[Schmutz '94]**	Conjecture: Bolza surface	Conjecture: Bolza surface
genus 3	Conjecture: Picard curve	Conjecture: Picard curve	Conjecture: Klein quartic	Klein quartic **[Fortier Bourque –P. '24+]**
higher genus	Local maximizers **[Schmutz '99]** **[Hamenstädt '01]** **[Fortier Bourque –Rafi '22]**			



The Bolza surface



The Klein quartic

The systole:

Facts: the moduli space $\mathcal{M}_g = \left\{ \begin{array}{c} \text{closed orientable hyperbolic} \\ \text{surfaces of genus } g \end{array} \right\} \Big/ \text{isometry}$ is a $(6g-6)$-dimensional orbifold.

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Lemma: Let $X \in \mathcal{M}_g$. Then

$$\mathrm{sys}(X) \leq 4 \cdot \mathrm{arcsinh}(\sqrt{g-1}) \quad \overset{g \to \infty}{=} \quad 2\log(g) + 2.772588\ldots + o(1)$$

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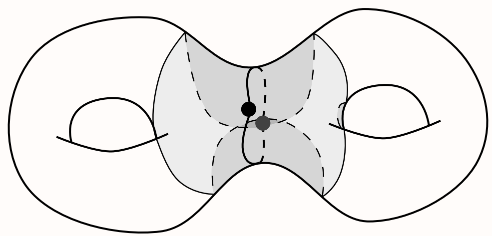

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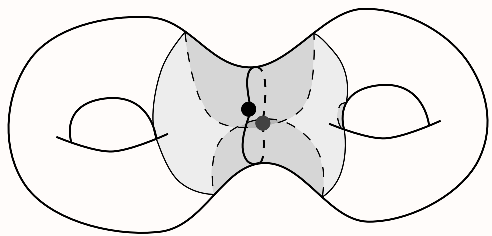

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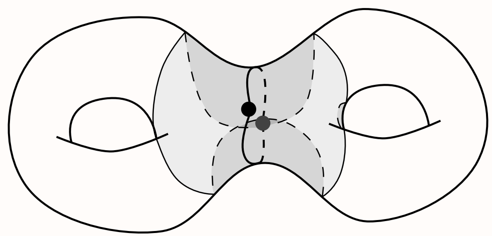

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$$2\pi(\cosh(\mathrm{sys}(X)/2) - 1) = \mathrm{area}(D_{\mathrm{sys}(X)/2}(x)) \leq \mathrm{area}(X) = 4\pi(g-1).$$

□

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[**Bavard '96**]:

$$\mathrm{sys}(X) \leq 2\mathrm{arccosh}\left(\frac{1}{2\sin(\pi/(12g-6))}\right) \quad \overset{g\to\infty}{=} \quad 2\log(g) + 2.68353\ldots + o(1)$$

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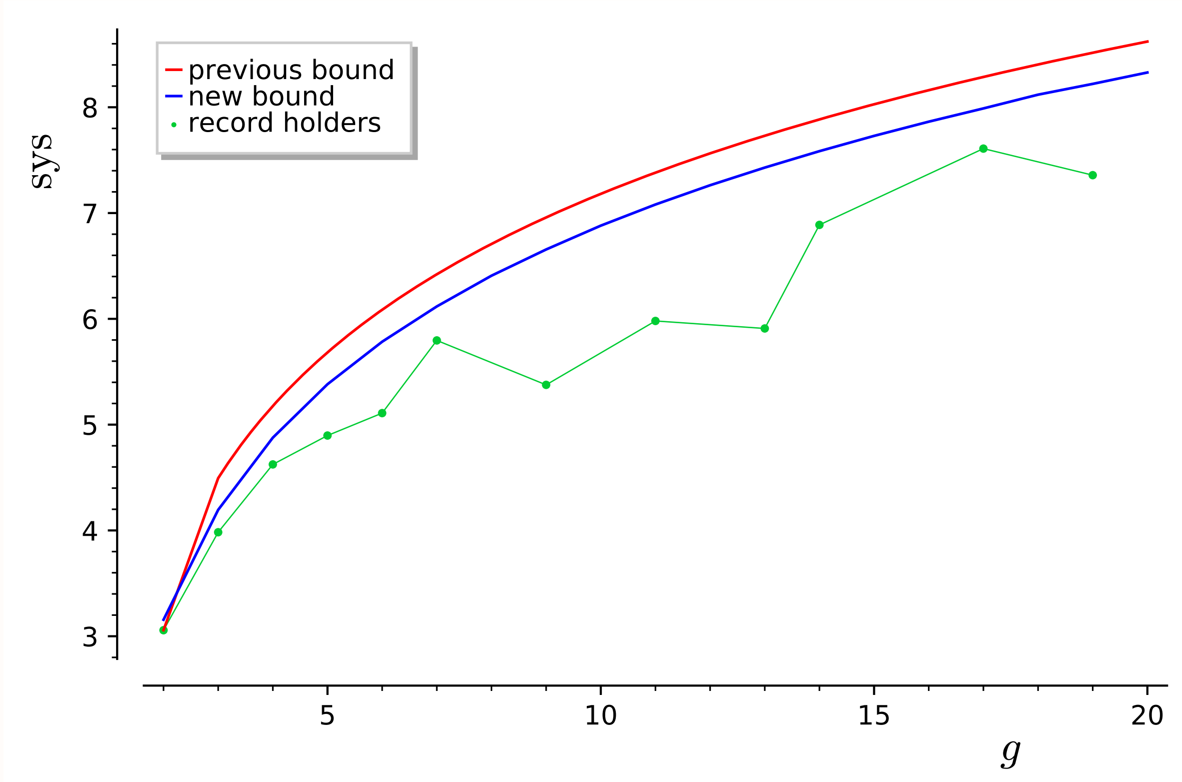

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[Katz–Sabourau '24]

$$\liminf_{g\to\infty} \frac{\max\{\mathrm{sys}(X);\ X \in \mathcal{M}_g\}}{\log(g)} \geq \frac{19}{120}$$

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Based on random triangulations combined with ideas inspired by graph theory [**Linial–Simkin ’21**]. □

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Open problem: Is a similar statement true for symmetric groups?

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which tends to 0 when

$$R \leq (1-\varepsilon)\cdot\log(p) \approx (1-\varepsilon)\cdot\log(\#G_p)/3 \approx (1-\varepsilon)\cdot\log(\mathrm{genus}(X_p))/3$$

□

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Lemma: If $X \in \mathcal{M}_g$, then
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Theorem [Budzinski–Curien–P. '21]:

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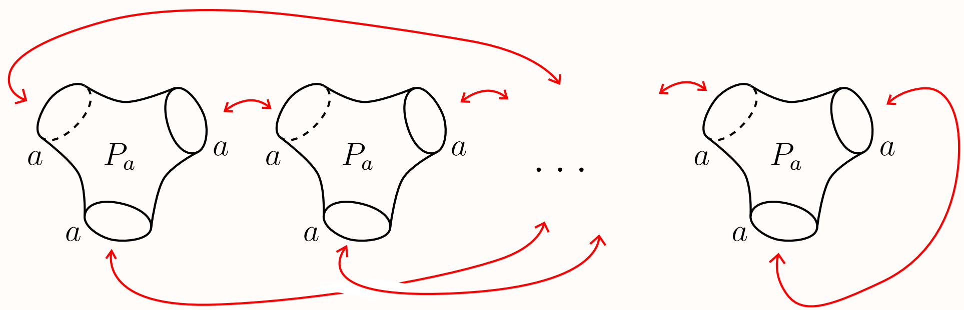
Proof sketch: A random construction:



$S_{g,a}$: random gluing of $2g - 2$ copies P_a with twist 0.

Goal: For every $\varepsilon > 0$, there exists an $a > 0$ such that:

$$\mathbb{P}\Big(\operatorname{diam}(S_{g,a}) \leq (1+\varepsilon)\cdot\log(g)\Big) \quad \xrightarrow{g\to\infty} \quad 1.$$

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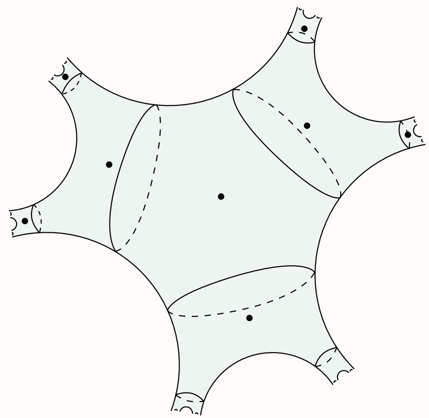

Two inputs:

- (Probabilistic) around "most" copies of P_a, $S_{g,a}$ "looks like" T_a up to depth $\approx \sqrt{g}$

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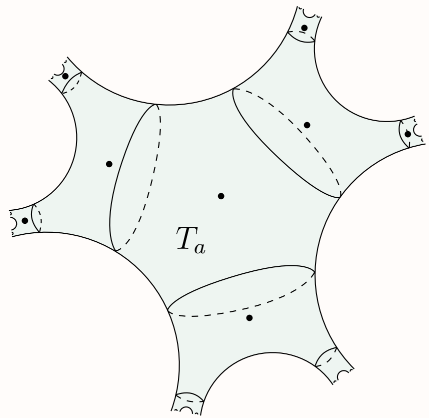


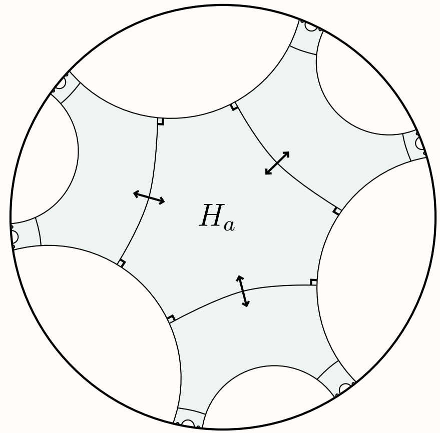
Two inputs:

- (Probabilistic) around "most" copies of P_a, $S_{g,a}$ "looks like" T_a up to depth $\approx \sqrt{g}$
- (Geometric) $m_0 \in T_a$ a midpoint, control exponential growth of

$$N_a(R) = \#\{m \in T_a \text{ midpoint};\ d(m, m_0) \leq R\}$$

as $R \to \infty$.

Geometry: Γ_a reflection group generated by the reflections in the sides of length $a/2$ of H_a



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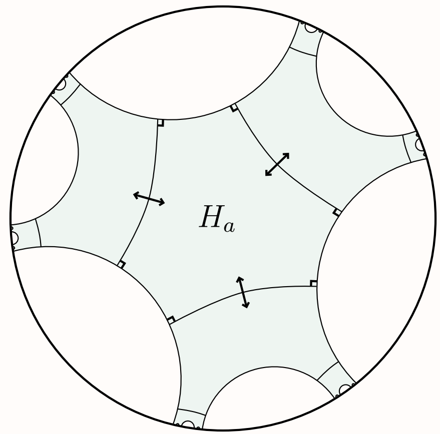

[Patterson '88, McMullen '98]

$$\#\Big(\Gamma_a \cdot x_0 \cap B(0,R)\Big) \sim \text{cst.}_a e^{\delta_a R} \quad \text{as } R \to \infty$$

and $\delta_a \to 1$ as $a \to \infty$.

Finishing the proof:

- Given any two pairs of pants $P, P' \in S_{g,a}$, with high probability, there are $\gg g^{1/2+\varepsilon}$ distinct pairs of pants at distance $\lesssim \frac{1}{2\delta_a} \log(g)$

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- Sum over the $\leq g^2$ pairs of paris of pants.

Thank you for your attention!