# Random Manifolds from Coloring

Chaim Even-Zohar

Joint work with Joel Hass

# Random Colors Model

**A growing self-avoiding walk in three dimensions and its relation to percolation**. RM Bradely, PN Strensky, JM Debierre, *Physical Review A*, 1992.

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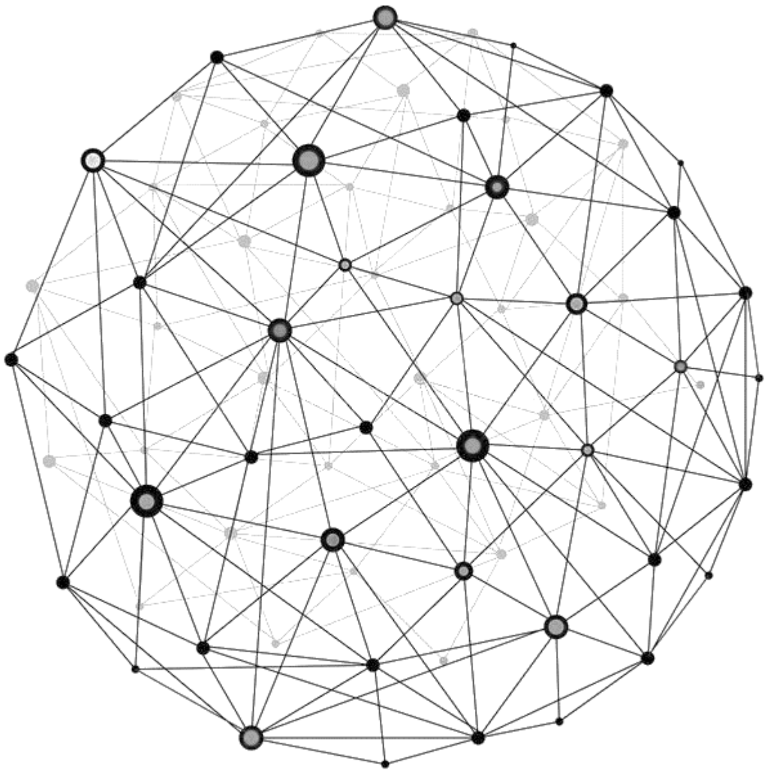

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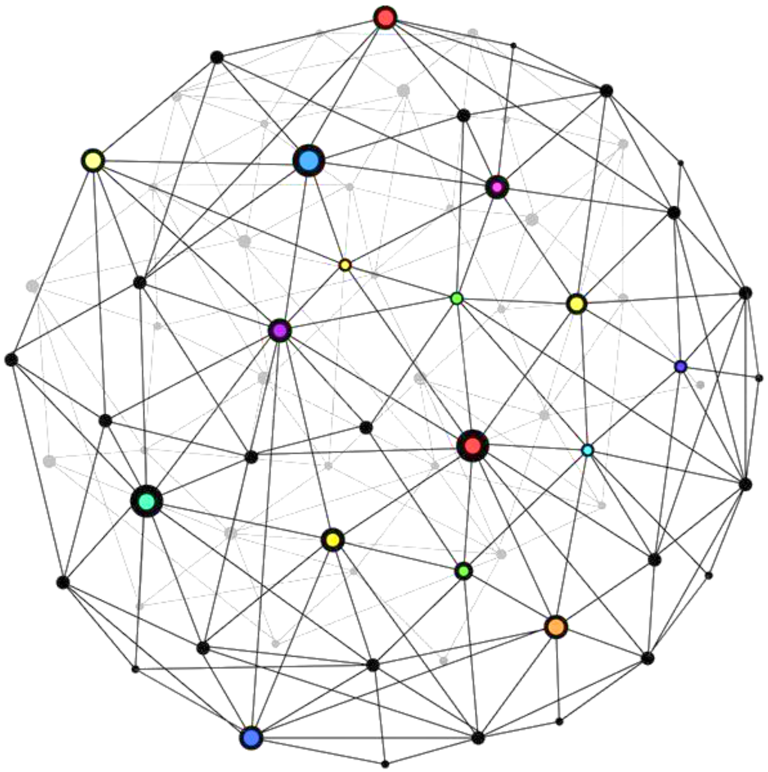
2. Randomly color vertices with k colors



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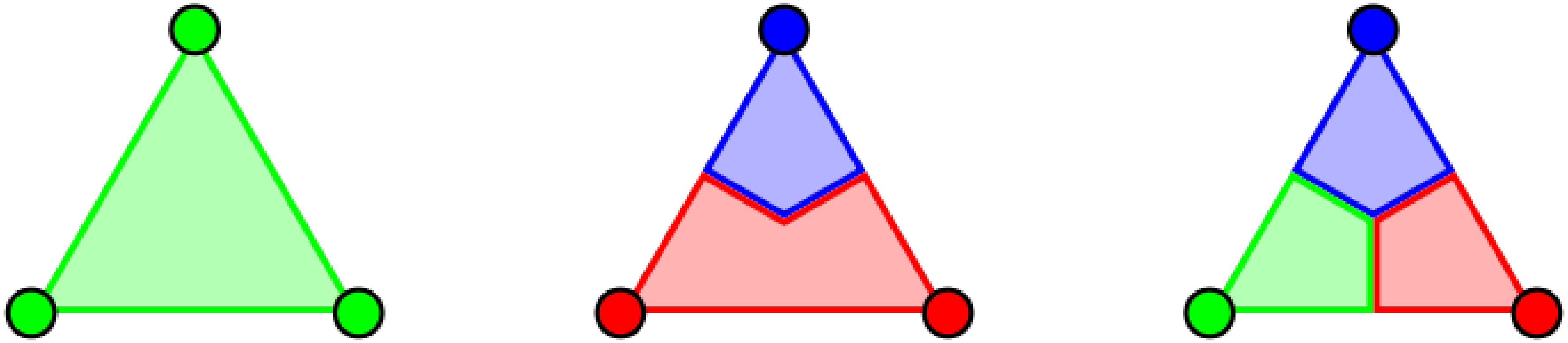
3. Assign to points the closest color(s)

= Voronoi Coloring



# Random Colors Model

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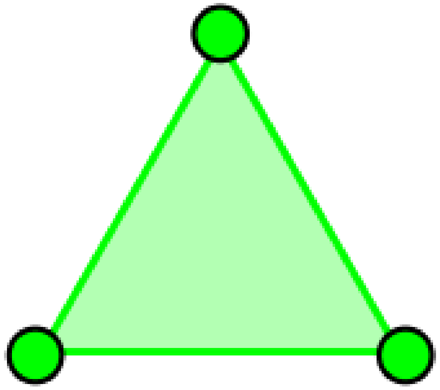
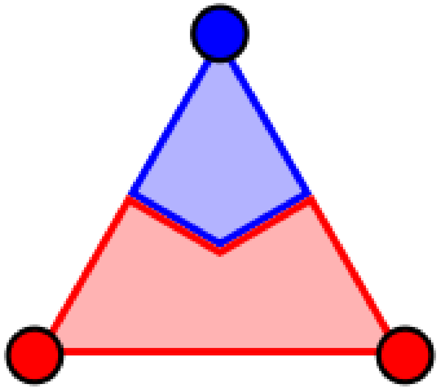
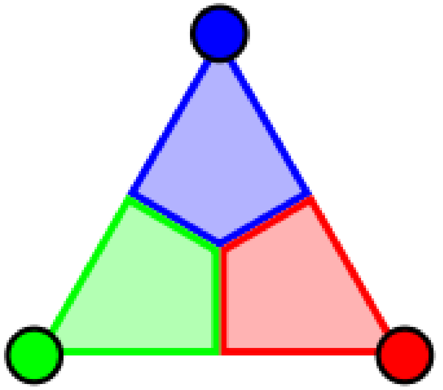
3. Assign to points the closest color(s)
4. Look at r-color classes, $1 \leq r \leq k$





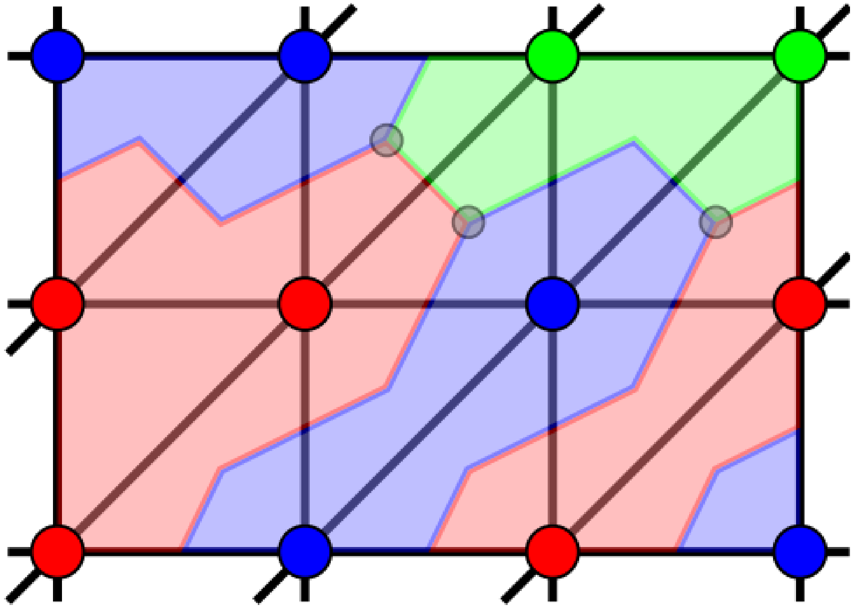


# Random Colors Model

d-dimensional

manifold M

k colors



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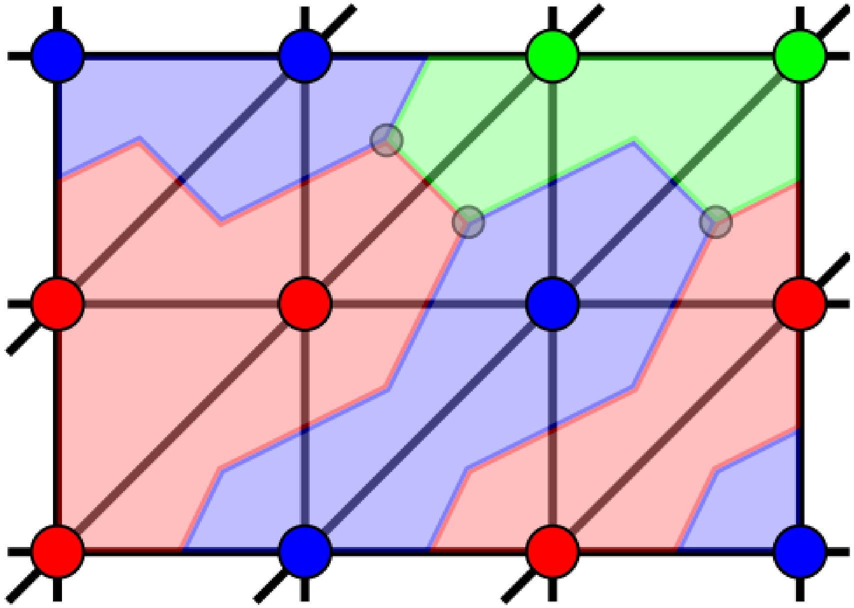

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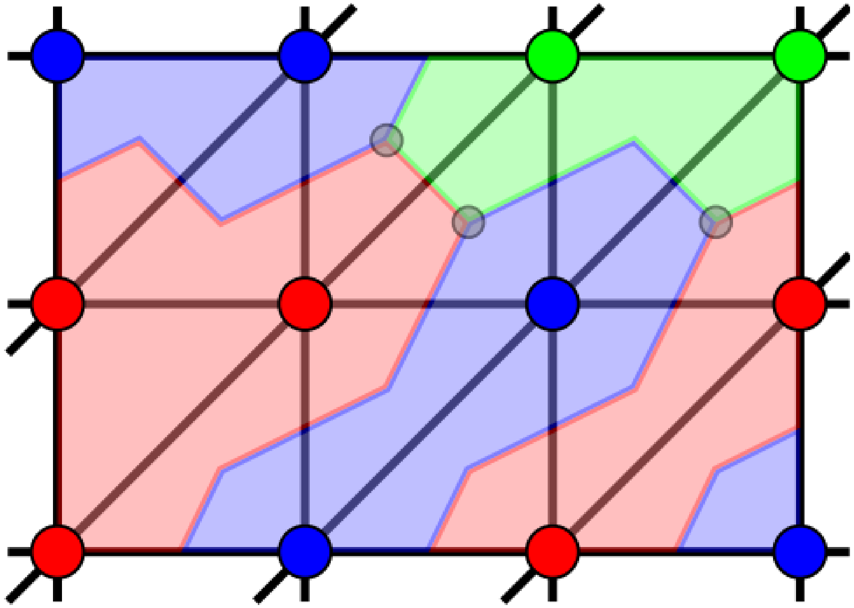

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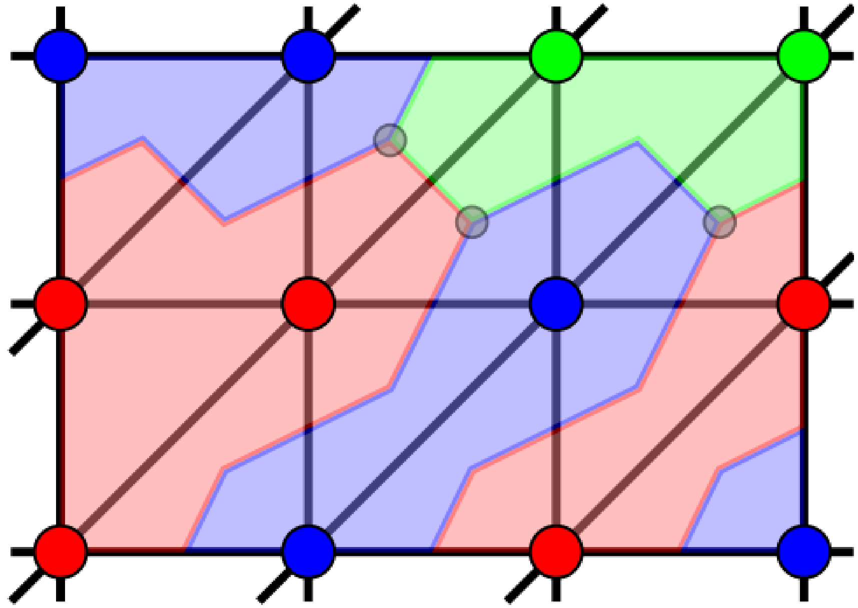
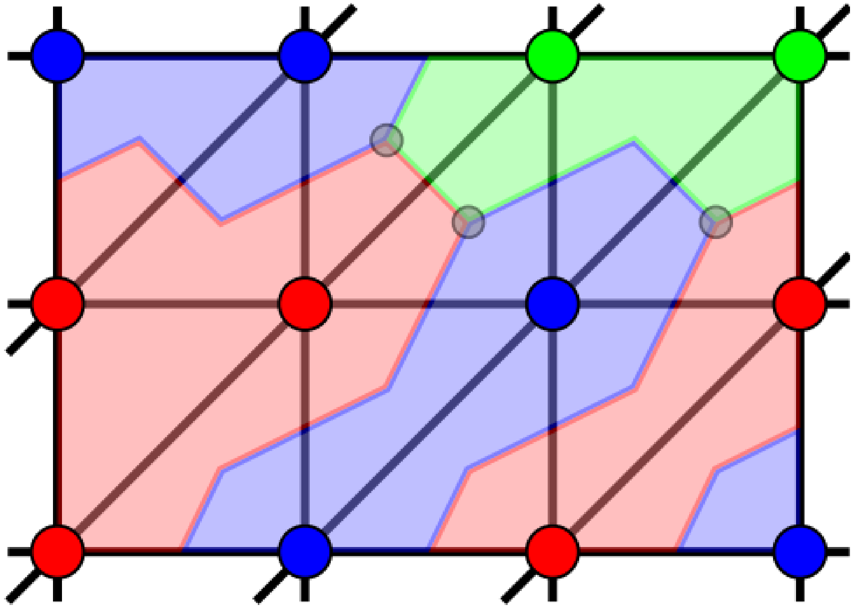


- 1-color class: d-dim
- 2-color class: (d-1)-dim
- r-color class: (d-r+1)-dim, $1 \leq r \leq k$

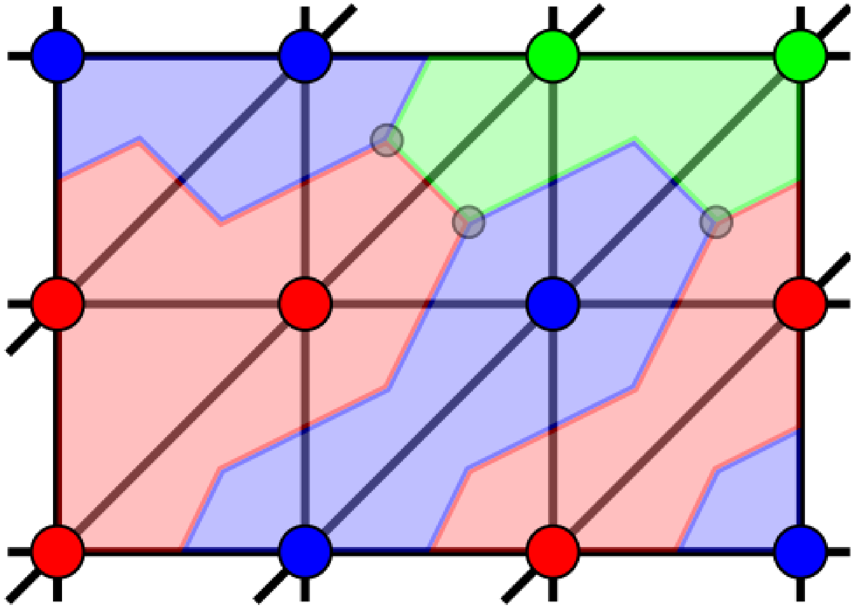
# Basic Questions

- Is a k-color class always a manifold?



# Basic Questions

- Is a $k$-color class always a manifold?
- Do all submanifolds of $M$ occur?



# Basic Questions

- Is a k-color class always a manifold?
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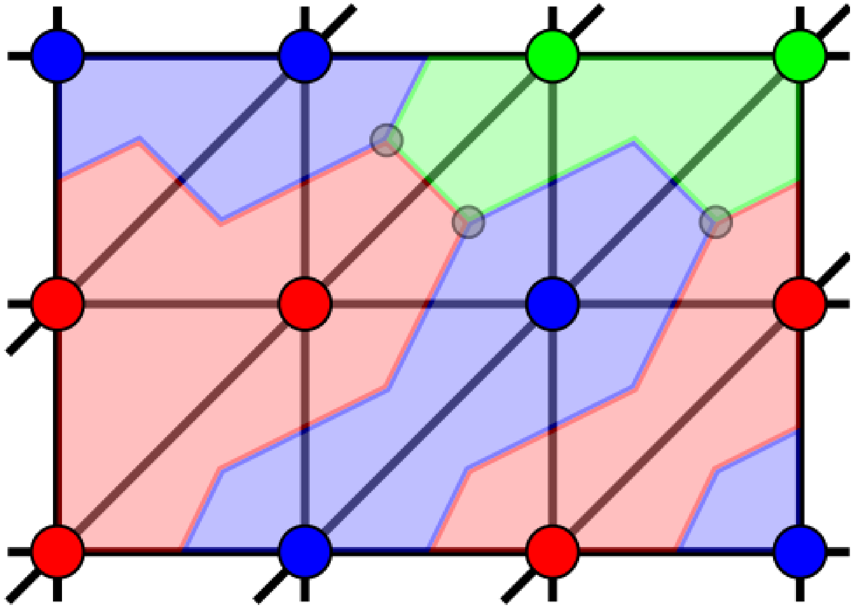
- Does a given manifold N occur in some M?



# Color classes are manifolds

**Theorem** E, Hass 2022

Let $X$ be the $k$-color class inside a combinatorial $d$-manifold $M$, colored using $k$ colors, for $2 \leq k \leq d$.

Then $X$ is a union of simplices that form a proper combinatorial $(d-k+1)$-dimensional submanifold of $M$.

For a subset of $m \leq k$ colors, the $m$-color class is a $(d-m+1)$-dim submanifold with boundary.

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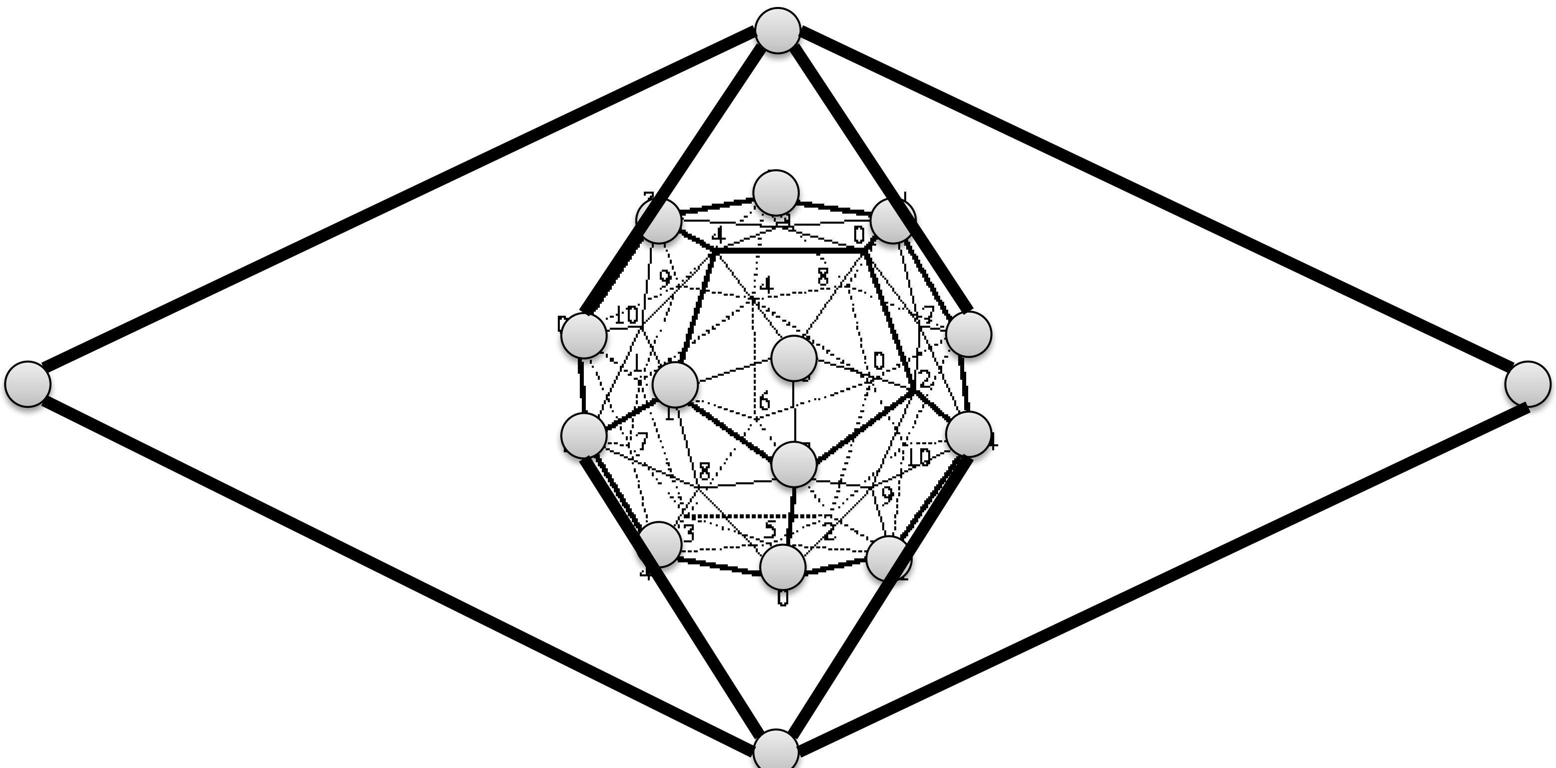

Double suspension of Poincaré Homology Sphere

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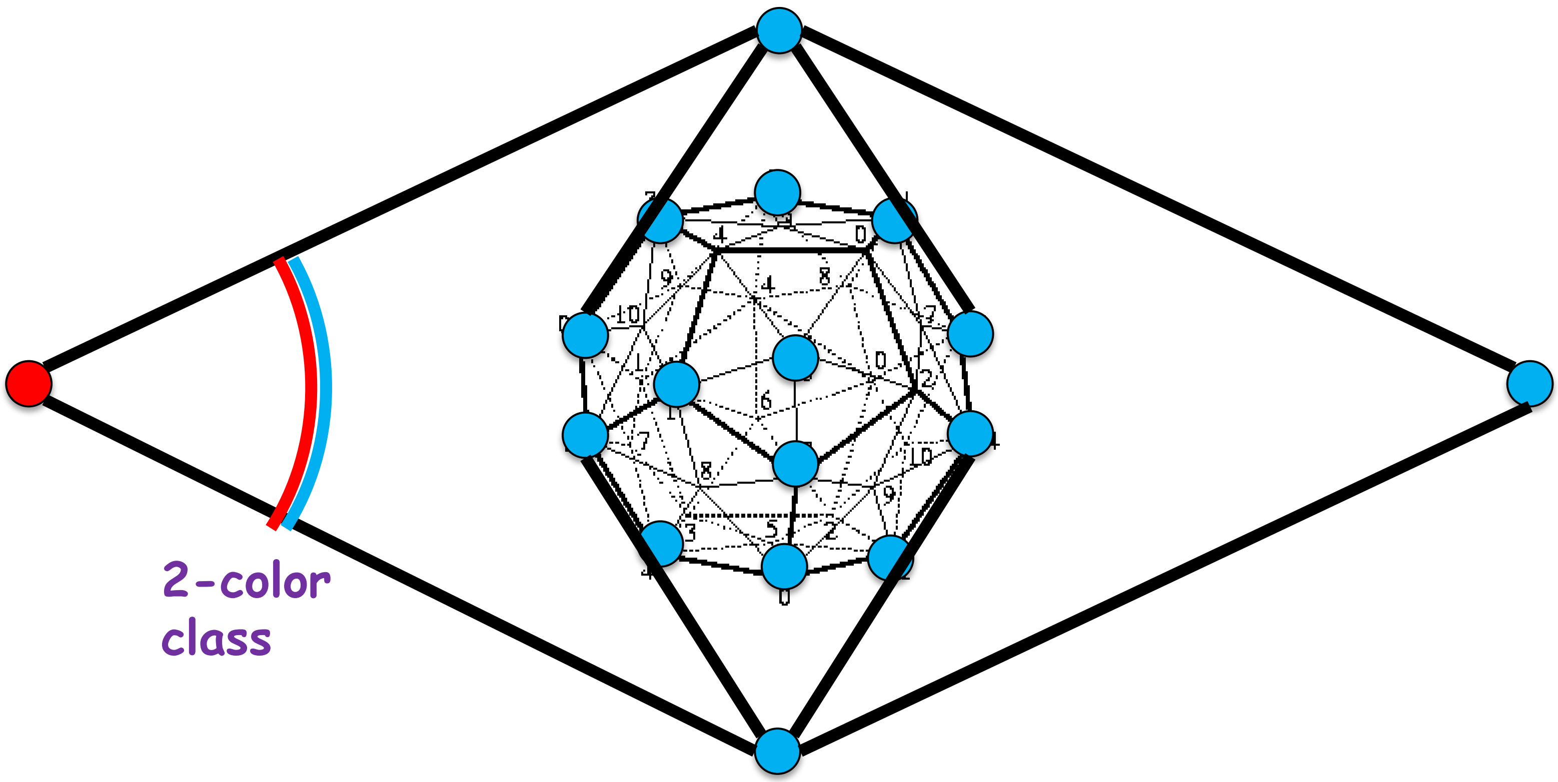


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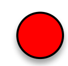
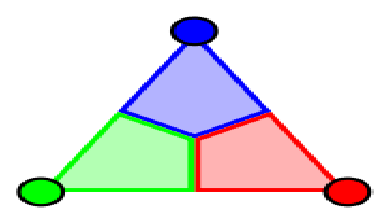
**1**-color class in **N x** 

**2**-color class in **N x** 

**3**-color class in **N x** 

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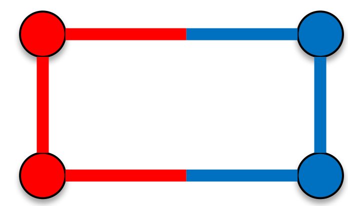
Yes, **twice** as the

2-color class in $M = N \times$ 

3-color class in $M = N \times \ldots$

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**Example:** $N = RP^2$ does not arise in any $M$!

**Why?** $RP^2$ is not null-cobordant, but $k$-color class = boundary[$(k-1)$-color class]

# Which Manifolds Arise?

**Theorem** E, Hass 2022

Let **k≥2**. A manifold **N** occurs as the entire **k**-color class in some closed **k**-colored manifold **M**, **if and only if N** is null-cobordant.

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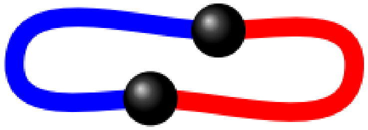

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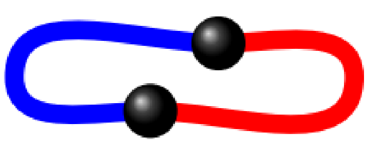

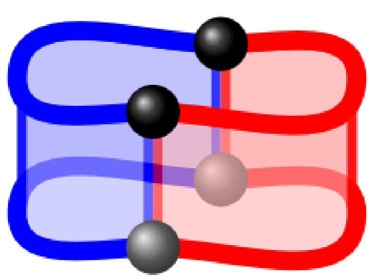

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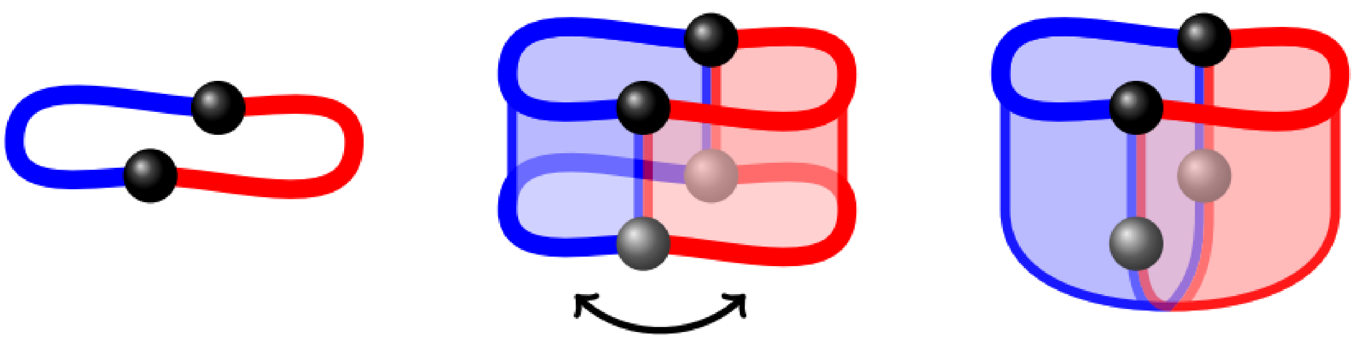

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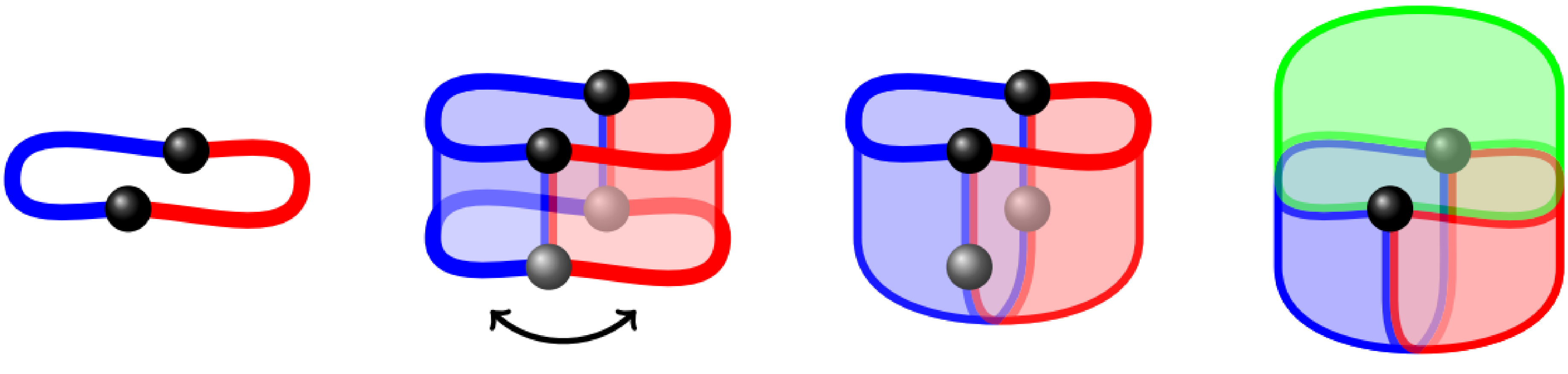

# Does N occur in M ?

**Another question**

Given a manifold **M** and **k≥2**, which manifolds **N** of codimension **(k-1)** arise from some **triangulation** and **k-coloring** of **M**?

# Does N occur in M ?

**Obstruction**: N must be embeddable in M

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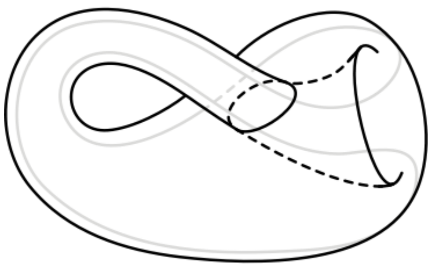
**Example**: no  in 3-colored $S^4$

Klein bottle

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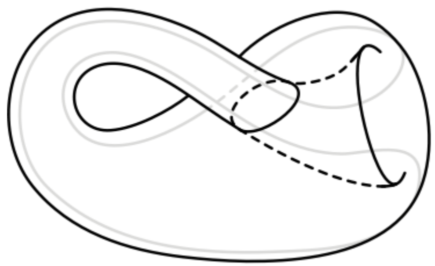
**Example:** no  in 3-colored $S^4$

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**Why?**

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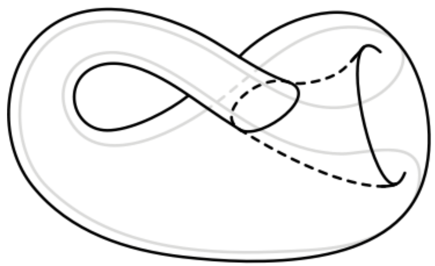
**Example:** no  in 3-colored $S^4$

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**In general:** $w_j(M)=0$ => Also $w_j(N)=0$

jth Stiefel-Whitney class: $w_j(\ldots) \in H^j(\ldots, Z_2)$

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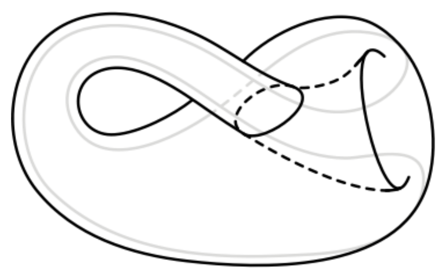
**Example:** no  in 3-colored $S^4$

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**In general:** $w_j(M)=0$ => Also $w_j(N)=0$

**Example:** no $CP^2 \# \overline{CP^2}$ in 6-colored $S^9$

# Does N occur in M ?

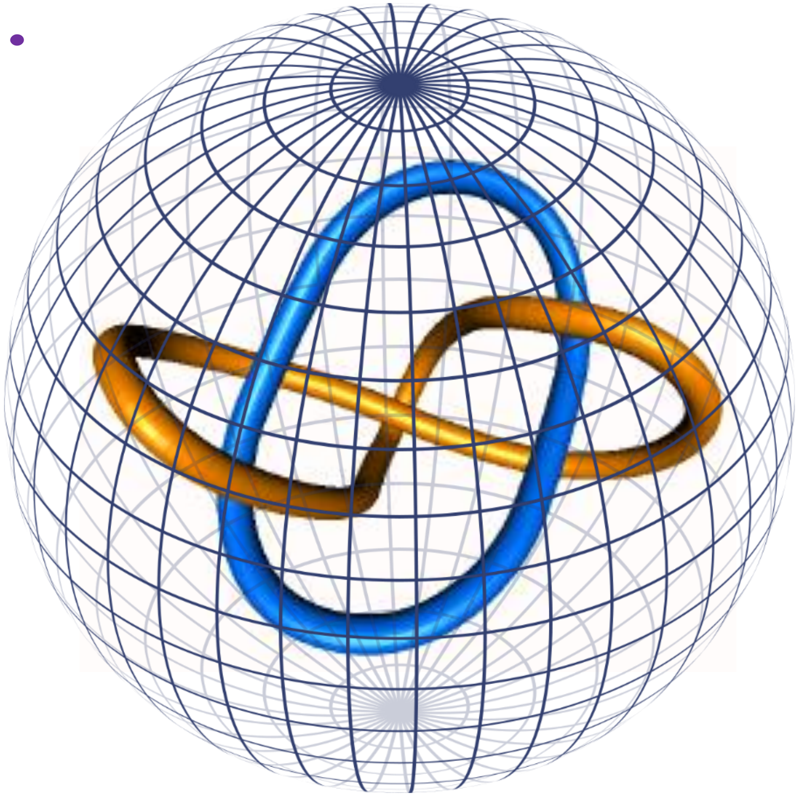
**Yet another question**

Given an embedding $N \subset M$ of codimension $k-1$, can it be obtained up to ambient isotopy from some triangulation and $k$-coloring of $M$?

# Does N occur in M ?

**Theorem** E, Hass 2022

Every link **L** is obtained as the **3**-color class of some triangulated $\mathbf{S^3}$.



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**Moreover**, letting $S^3 = \partial B^4$, every surface in $B^4$ bounded by L is obtained by 3-coloring $B^4$.

**Generally**, knotted $S^{d-2}$ in $S^d = \partial B^{d+1}$ bounding an orientable $F^{d-1}$ are realized by 3-colorings.

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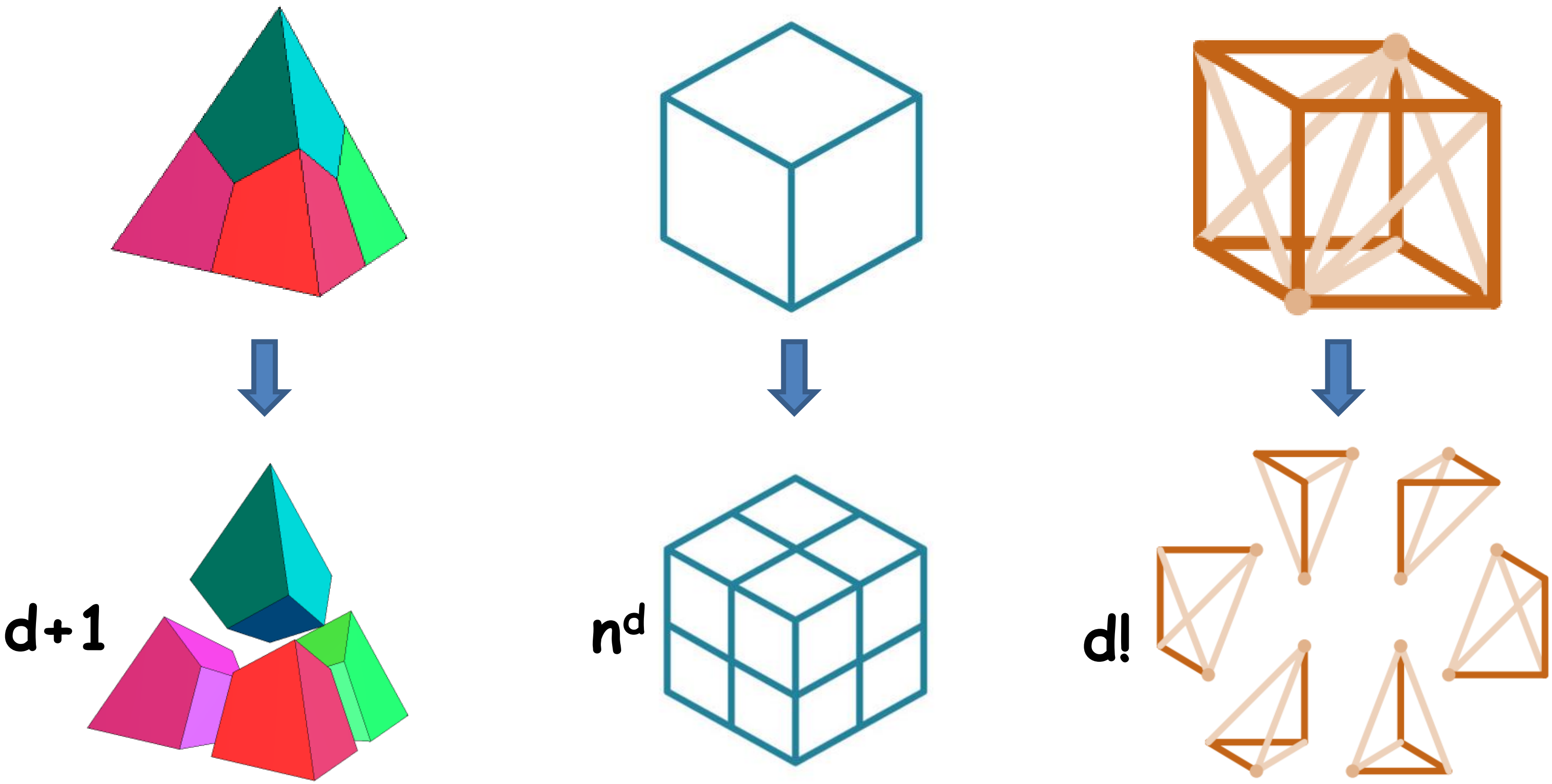
- Subdivide **M** according to a parameter **n**



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- Start with a triangulated manifold **M**
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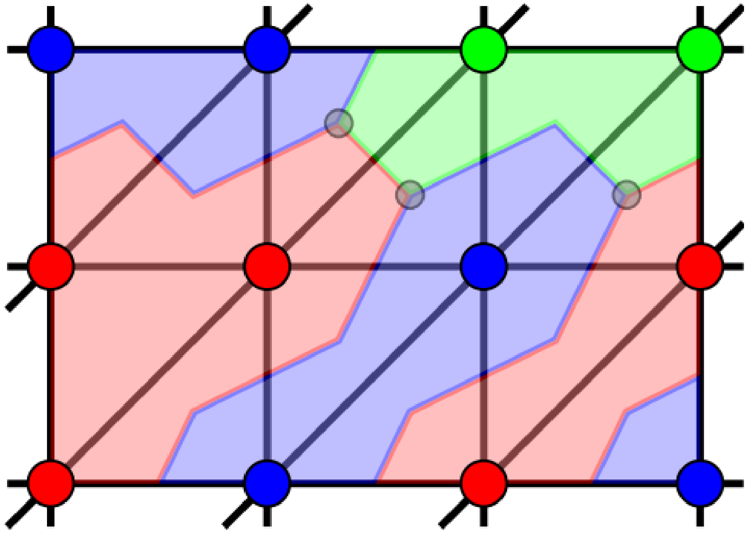
- Assign to vertices iid colors **{1,2,…,k}** distributed **p = ($p_1$,$p_2$,…,$p_k$)**



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- **Variations**: location-dependent **p**, fixed boundary conditions, recoloring, etc.

# THANK YOU!