Computing Character Varieties and Schemes in $\mathrm{SL}_2(\mathbb{C})$

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BIRS

Knot Theory Informed by Random Models and Experimental Data

Variety/Scheme of Representations

- $\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid r_1, \ldots, r_m \rangle$ finitely presented group (eg $\Gamma = \pi_1(S^3 \setminus K)$).

$$\hom(\Gamma, \mathrm{SL}_2(\mathbb{C})) \subset \mathrm{SL}_2(\mathbb{C}) \times \cdots \times \mathrm{SL}_2(\mathbb{C}) \subset \mathbb{C}^{4n}$$

  It is an algebraic subset of $\mathbb{C}^{4n}$ called the variety of representations.
- It has more structure than a variety, it is an affine scheme (with perhaps several components and multiple points, eg $x^2 = 0$ is different from $x = 0$ as a scheme).
- $\mathrm{SL}_2(\mathbb{C})$ acts on $\hom(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ by conjugation
  The topological quotient $\hom(\Gamma, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ may be non-Hausdorff.

Def The character of $\rho \in \hom(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is the map

$$\begin{array}{rcl} \chi_\rho \colon \Gamma & \to & \mathbb{C} \\ \gamma & \mapsto & \mathrm{trace}(\rho(\gamma)) \end{array}$$

Lemma $\overline{\mathrm{Orbit}(\rho_1)} \cap \overline{\mathrm{Orbit}(\rho_2)} \neq \emptyset$ iff $\chi_{\rho_1} = \chi_{\rho_2}$.

Thm (Procesi) $X(\Gamma) = \{\chi_\rho \colon \Gamma \to \mathbb{C} \mid \rho \in \hom(\Gamma, \mathrm{SL}_2(\mathbb{C}))\}$ is the "algebraic quotient".

$$X(\Gamma) = \hom(\Gamma, \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2(\mathbb{C}))$$

Scheme of characters

- $\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid r_1, \ldots, r_m \rangle$ finitely presented group (eg $\Gamma = \pi_1(S^3 \setminus K)$).
- What is the algebraic structure of $X(\Gamma) = \{\chi_\rho \colon \Gamma \to \mathbb{C} \mid \rho \in \mathrm{hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))\}$?

Def: Trace function:

$$\begin{array}{rcl} t_\gamma \colon \mathrm{hom}(\Gamma, \mathrm{SL}_2\mathbb{C}) & \to & \mathbb{C} \\ \rho & \mapsto & \mathrm{trace}(\rho(\gamma)) \end{array}$$

(Procesi) $X(\Gamma)$ embeds in $\mathbb{C}^N$ with coordinates $t_{\gamma_1}, \ldots, t_{\gamma_N}$ for some $\gamma_1, \ldots, \gamma_N \in \Gamma$, as an algebraic subset (as an affine scheme) called the scheme of characters

- Procesi proves that the algebral of $\mathrm{SL}_2(\mathbb{C})$-invariant polynomial functions $\mathbb{C}[\mathrm{hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))]^{\mathrm{SL}_2(\mathbb{C})}$ is finitely generated by $t_{\gamma_1}, \ldots, t_{\gamma_N}$
  and define $X(\Gamma)$ by the property $\mathbb{C}[X(\Gamma)] = \mathbb{C}[\mathrm{hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))]^{\mathrm{SL}_2(\mathbb{C})}$.
  Then points in $X(\Gamma)$ are viewed as a characters.

Scheme of characters for a free group

Thm (Fricke-Klein) For $F_2 = \langle a, b \mid \rangle$, there is an isomorphim

$$(t_a, t_b, t_{ab}) : X(F_2) \xrightarrow{\cong} \mathbb{C}^3$$

- Equivalently the function algebra is polynomial $\mathbb{C}[X(F_2)] = \mathbb{C}[t_a, t_b, t_{ab}]$
- or for every $\gamma \in \Gamma$, the trace function t_γ is a unique polynomial on t_a, t_b and t_{ab}

- The proof uses trace identities for $A, B \in \mathrm{SL}_2(\mathbb{C})$:
  - $\mathrm{trace}(AB) = \mathrm{trace}(BA) \rightsquigarrow t_{\gamma\mu} = t_{\mu\gamma}$
  - $\mathrm{trace}(A^{-1}) = \mathrm{trace}(A) \rightsquigarrow t_{\gamma^{-1}} = t_\gamma \qquad \forall \gamma, \mu \in \Gamma.$
  - $\mathrm{trace}(AB) + \mathrm{trace}(AB^{-1}) = \mathrm{trace}(A)\mathrm{trace}(B) \rightsquigarrow t_{\gamma\mu} + t_{\gamma\mu^{-1}} = t_\mu t_\gamma$

  Use the identities to reduce the word length of γ in t_γ.
  Eg: $t_{a^2} = t_a^2 - 2, \qquad t_{aba^{-1}b^{-1}} = t_a^2 + t_b^2 + t_{ab}^2 - t_a t_b t_{ab} - 2$
- For F_n in general $X(F_n)$ is also known (Magnus 1980).
  The affine scheme $X(F_n)$ is irreducible and has no multiple points (eg a variety).

Goal: From a finite presentation of Γ, what is the role of the relations?
Find an algorithm to compute $X(\Gamma)$

Main result

- $\Gamma = \langle \gamma_1, \ldots, \gamma_n \mid r_1, \ldots, r_m \rangle$ finitely presented group (eg $\Gamma = \pi_1(S^3 \setminus K)$).

Thm (Fico-Montesinos 1993) As variety, $X(\Gamma)$ is the subvariety of $X(F_n)$ with equations

$$t_{r_i} = 2, \qquad t_{r_i\gamma_j} = t_{\gamma_j}$$

for $1 \leq i \leq m$, $1 \leq j \leq n$ *(and taking the radical ideal)*.

Thm (Heusener-P 2023) As scheme, $X(\Gamma)$ is the subscheme of $X(F_n)$ with equations

$$t_{r_i} = 2, \qquad t_{r_i\gamma_j} = t_{\gamma_j}, \qquad t_{r_i\gamma_j\gamma_k} = t_{\gamma_j\gamma_k}$$

for $1 \leq i \leq m$, $1 \leq j < k \leq n$.

- Steps of the proof:
  1. As subscheme of $X(F_n)$, $X(\Gamma)$ is given by the equations $t_\gamma = t_\mu$ for every pair $\gamma, \mu \in F_n$ that project to the same element in Γ (using the skein algebra).
  2. Use trace relations to reduce to these equations.
- ...BUT COMPUTATIONALLY INEFFICIENT

Why care about schemes?

Ex: $S^3(K_8, 3)$ orbifold with underlying space S^3, branching locus the figure eight knot and ramification index 3.

$$\Gamma = \pi_1^{\mathrm{orb}}(S^3(K_8, 3)) = \langle a, b \mid ab^{-1}a^{-1}ba = bab^{-1}a^{-1}b,\ a^3 = 1\rangle$$

- $X(\Gamma)$ embeds in $\mathbb{C}^2$ with coordinates $t_a = t_b$ and t_{ab}. It has two simple points and one double point:

$$t_a - 2 = t_{ab} - 2 = 0, \quad t_a + 1 = t_{ab} + 1 = 0 \quad \text{and } t_a + 1 = (t_{ab} - 1)^2 = 0$$

- The orbifold $S^3(K_8, 3)$ is Euclidean and the double point is a lift to $\mathrm{SU}(2)$ of the rotational part of the holonomy in $\mathrm{SO}(3) \cong \mathrm{PSU}(2)$

Ex: $S^3(Wh, (8, 4))$ orbifold with underlying space S^3, ramification locus the Whitehead link and ramification indexes 8 and 4.

- $X(\pi_1^{\mathrm{orb}}(S^3(Wh, (8, 4))$ has 21 simple points and 2 triple points.
- The orbifold $S^3(Wh, (4, 2))$ has a Nil structure and its holonomy is related to triple points.

Thanks you for your attention