Knots in Self-Avoiding Polygons

Neal Madras
Department of Mathematics and Statistics
York University
Toronto, Canada

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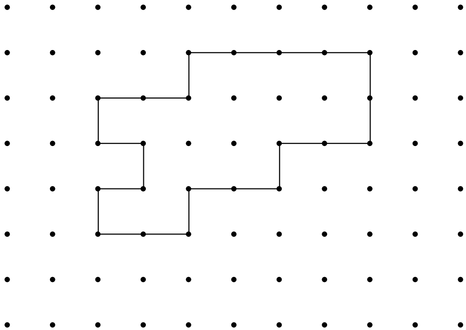
Self-avoiding polygons

An N-step self-avoiding polygon (SAP) is a simple closed curve consisting of N edges of the lattice $\mathbb{Z}^d$ ($d \geq 2$).



This is a 22-step self-avoiding polygon in $\mathbb{Z}^2$.

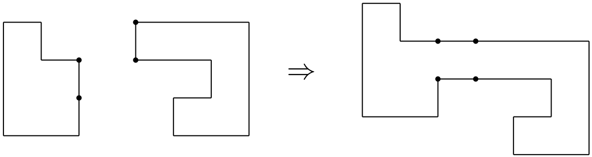
Let p_N be the number of N-step SAPs modulo translation.

E.g. in $\mathbb{Z}^2$: $p_4 = 1$, $p_6 = 2$, $p_8 = 7$

in $\mathbb{Z}^3$: $p_4 = 3$, $p_6 = 22$, $p_8 = 207$

Let p_N be the number of N-step SAPs modulo translation.

Concatenation: $$p_{N+M} \geq p_N \, p_M \qquad \forall M, N \text{ (even)}.$$



(may need a rotation in 3 or more dimensions)

$$\therefore \quad \lim_{N\to\infty} p_N^{1/N} = \mu := \sup_N p_N^{1/N}. \qquad \text{That is, } p_N = \mu^{N+o(N)}.$$

- Motivation: model of conformations of ring polymer molecules

Knots in Self-Avoiding Polygons

Let K be a knot type (e.g. trefoil or unknot).

Let $p_N[K]$ be the number of N-step SAP's in $\mathbb{Z}^3$ of knot type K.

What we know about the asymptotics of $p_N[K]$:

(i) For the unknot O: $\mu[O] := \lim_{N\to\infty} p_N[O]^{1/N}$ exists.
(Proof: $p_n[O]\, p_m[O] \le p_{n+m}[O]$.)

(ii) $\mu[O] < \mu$. More generally, for any fixed knot type K, $p_N[K]$ is exponentially smaller than p_N.
(Sumners & Whittington 1988; Pippenger 1989)

Open Problem: For $K \neq O$, prove that
$\mu[K] := \lim_{N\to\infty} p_N[K]^{1/N}$ exists and equals $\mu[O]$.

(Easy part: $p_n[O]\, p_m[K] \le p_{n+m}[K]$. Fixing m shows
$\mu[O] \le \mu[K]$, if the limit exists.)

Simulations and theoretical arguments (Orlandini et al. 1996) indicate that

$$\frac{p_N[K]}{p_N[O]} \asymp N^{f(K)} \quad \text{as } N \to \infty$$

where $f(K)$ is the number of prime knots in the knot K.

Can we prove this for knots in a slab between two planes, e.g. $\mathbb{Z} \times \mathbb{Z} \times [0, 10]$? ...still too hard.

How about for knots in an infinitely long tube, e.g. $\mathbb{Z} \times [0, 10] \times [0, 10]$? ...still too hard.

How about for knots in the narrowest possible tube, i.e. $\mathbb{Z} \times [0, 2] \times [0, 1]$? ...YES!
(iii) Beaton, Ishihara, Atapour, Eng, Vazquez, Shimokawa, Soteros 2022 arxiv:2204.06186 (44 pages)

(From the abstract: "As part of the proof, we show that a 4-plat diagram of a 2-bridge link can always be unknotted by the insertion of a 4-braid diagram whose crossing number is bounded by the minimal crossing number of the link.")

Here is one more thing that we know:

(iv) The commonest knot type is not exponentially rare:

For each N, let K_N be the knot type K that maximizes $p_N[K]$. Then

$$\lim_{N\to\infty} p_N[K_N]^{1/N} = \mu.$$

(Madras 2023)