

RANDOM KNOTS FROM TURBOKNOTS

T. Wolf, Brock University

April 2, 2023

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Knot Colouring

HOMFLY-PT Computations

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One of its strength is its ability to simplify diagrams which is useful, for example, when computing knot invariants.

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- ▶ Questions are announced 2 weeks before each contest to study 'Food for Thought'
- ▶ May 2023 contests featured an interactive **knot colouring challenges** for 14,800 students of grades 7 to 12: **grade 7/8** 44%, **grade 9/10** 39%, **grade 11/12** 24%

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Visualization, Scope


















Copying the Terminal: `y`

Zooming in/out: `Ctrl +/- z` (after resizing terminal)

Working with a stack of diagrams: `+ PgUp/Dn Del`

Shifting the visible Window: `S-↑, S-↓, S-←, S-→`

Generating and Simplifying Random Knot Diagrams

r R [Ctrl r]
l b kn/big/128 
d s S-1 
s S-2 
d S-1 
d S-2 
d S-4 
s S-3 
d S-3 
d S-4 
d S-1 
d S-5 
d S-4 
p P 
 [PgDn] []
l b  d m 

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Knot Colouring I

Computing the Smith Normal Form of coeff matrix gives colouring numbers.


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[1 b kn/bigprime/21](#) 

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
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→ $1049879229 = 3 \times 7 \times 23 \times 2173663$

$$3 = 3$$

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→ May be useful to visualize tangles or intertwined prime knots

Knot Colouring II

l b kn/bigprime/38 ↩

c c 1 ↩ ↩

Knot Colouring II

l b kn/bigprime/38 

c c 1  

→ 18167191515 = 3 × 3 × 3 × 3 × 3 × 5 × 11 × 1359311

Knot Colouring II

l b kn/bigprime/38 ↩

c c 1 ↶ ↷

→ $18167191515 = 3 \times 3 \times 3 \times 3 \times 3 \times 5 \times 11 \times 1359311$

→ Colouring numbers 3 and $3^5 = 243$ have both multiplicity 1.

Knot Colouring III

l b kn/bigprime/39 ↵
c c 1 ↶ ↷

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


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Colour smaller knots visually: `n n 8_18`  

`c c 1` 

`c 2 3`  (colour diagram) `2`  

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Display of the growth of colouring invariants: [a o](#) ↩

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- ▶ direct HOMFLY-PT polynomial substitution of the Hopf Link and of knots $3_1, 4_1, 5_1, 5_2$.

HOMFLY-PT computations II

File name in bigprime/	# of crossings	TurboKnots time	Regina time
23	49	2 s	0 s
47	54	11 s	0 s
38	64	21 s	0.5 s
21	68	1:10 min	0.5 s
20	83	5:20 min	21 s

REGINA is faster for larger prime knots

HOMFLY-PT computations III

File name in composed/	# of crossings	TurboKnots time	Regina time
pktest	78	0 s	0 s
pk0	6	0 s	0 s
pk1	21	0 s	0 s
pk2	25	0 s	0 s
pk3	38	0 s	0.4 s
pk4	38	0 s	0.6 s
pk5	52	0 s	3.8 s
pk6	78	12 s	21 s

TURBOKNOTS is slightly faster for composite knots

HOMFLY-PT computations IV

File name in ukn/0/	# of cross- ings	TurboKnots time	Regina time
TuzunSikora	21	0 s	0 s
SikoraTuzun	23	0 s	0 s
Gordian Gordian	141	0 s	-

These are unknots

HOMFLY-PT computations V

File name in big/	# of crossings	TurboKnots time	Regina time
98	754	0 s	2.2 s
0	794	0 s	1.3 s
48	816	0 s	1 s
102	832	0 s	1 s
44	867	0 s	1 s
86	869	0 s	2.4 s
76	873	0 s	2 s
38	888	0 s	1.4 s

TURBOKNOTS is faster for knots that can be heavily simplified.

HOMFLY-PT computations VI

... read from disk all 313230 knots with up to 15 crossings and compute and verify HOMFLYPT computation in 3:10 min,

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.. simplify the Gordian (un-)knot diagram with 141 crossings and compute the polynomial to 1 in about 1 ms.

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Computing Unknotting Numbers I

Usual definition: The unknotting number of a knot is n if there exists a diagram where n crossings switched (at the same time!) give the unknot.

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It had been conjectured by Bernhard-Jablan that a minimal unknotting sequence could be determined starting with some minimal crossing diagram for a knot. This has been shown to be false by Brittenham and Hermiller.

Computing Unknotting Numbers II

The 'unknotting number' computed in `TURBOKNOTS` is recursively defined. It is n if there exists a minimal diagram of that knot (with minimal number of crossings) that has at least one crossing which being switched results in a knot with unknotting number $n - 1$.

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The 'unknotting number' computed in `TURBOKNOTS` is recursively defined. It is n if there exists a minimal diagram of that knot (with minimal number of crossings) that has at least one crossing which being switched results in a knot with unknotting number $n - 1$.

This gives the correct value if for each knot at least one minimal diagram has a crossing which being switched lowers the unknotting number.

Computing Unknotting Numbers III

Procedure:

- ▶ Know all minimal diagrams for all knots. That can be many, e.g. knot $15n_{2100}$ has (at least) 3986.

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- ▶ Identify the prime knot or composite knot through its prime knots and look their unknotting number up in the database. Continue recursively if their unknotting number is not known yet.

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- ▶ Store the computed unknotting number in the database.

Computing Unknotting Numbers IV

Comments:

We also apply the unproven but commonly accepted hypothesis that the unknotting number of a composite knot is the sum of the unknotting numbers of its prime knots.

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As a by-product for each knot not only the unknotting number but also the minimal and the maximal number of simplifying crossings of all minimal diagrams are determined and stored.

Computing Unknotting Numbers V

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Example: $11n_{64}$ has 14 minimal diagrams. For 2 diagrams all switches result in knots with unknotting number 2, 6 diagrams have 2 switches which result in a knot with unknotting number 1 and 6 diagrams have a single switch resulting in a knot with unknotting number 1.

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Example: 10_{139} with $u = 4$ has a diagram where 2 simplifying switches each change the knot into 10_{161} with $u = 3$ which has a diagram where 2 simplifying switches each change the knot into 10_{145} with $u = 2$.

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TURBOKNOTS:

Download: `wget`

<https://cariboutests.com/games/knots/TurboKnots.tar.gz>

Unpack: `tar xzf TurboKnots.tar.gz`

Call: `./TurboKnots`

Help: `./TurboKnots '??'`

Colouring numbers and multiplicities:

<https://cariboutests.com/games/knots/colour3-15-N.txt>

HOMFLYPT polynomials:

<https://cariboutests.com/games/knots/HOMFLY3-15.txt>

Unknotting numbers:

<https://cariboutests.com/games/knots/uk3-15.txt>

Overview of the 3 data files:

<https://cariboutests.com/games/knots/readme.txt>

The End

Thank you!