Randomness and Quasiperiodicity in Mathematical Physics

David Damanik (Rice University),
Wencai Liu (Texas A&M University),
Constanza Rojas-Molina (CY Cergy Paris Universite),
Mira Shamis (Queen Mary Univerisity of London)

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1 Overview of the Field

The study of structure and randomness is of broad appeal in many areas in mathematics. One instance is given by models arising in mathematical physics, and a particular example is given by the Schrödinger equation, which describes the time evolution of a quantum state in a given environment, modeled by a suitable choice of potential function. Random media are naturally modeled by choices of potentials that are random in a suitable sense, whereas media displaying order are often modeled by suitable choices or periodic or quasiperiodic potentials.

The foundations of the mathematical theory for these types of Schrödinger equations were laid in the late 1970's and early 1980's. Since the early days of these investigations, it has been very beneficial to treat these two classes of interest under a common umbrella, the class of stationary or ergodic potentials, as this explains the common features in a very clear way and also provides a framework for mutually beneficial exchanges between the two subareas.

Nevertheless, the two subareas have been so active and successful that most conferences and workshops covering them are typically dedicated to only one of the subareas in order to accommodate the number of researchers and recent contributions in a comprehensive way.

The workshop aimed to catalyze interactions among experts, fostering the exchange of innovative ideas and methodologies. Recent years have witnessed significant advancements within these specialized fields. By connecting these areas with random matrix theory and statistical mechanics, the workshop set the stage for potentially groundbreaking progress in understanding random and quasiperiodic systems.

An innovative aspect of this workshop was the engagement of specialists in related fields like random matrix theory and statistical mechanics. These experts brought fresh perspectives, enriching the discussions with insights that have recently shown to deeply interconnect with random matrices and the statistical mechanics of diverse landscapes.

The BIRS workshop convened preeminent mathematicians in the study of random Schrödinger operators and quasiperiodic operators, alongside luminaries in random matrix theory and statistical mechanics. This meeting not only highlighted the ongoing contributions to these fields but also underscored the interdisciplinary synergy critical for pioneering future research directions.

2 Recent Developments and Open Problems

In random systems, several new developments have strengthened the interaction with physics in recent years, in particular, the study of entanglement entropy and the study of disordered quantum spin systems. Both subjects aim at a better understanding of the spectral and dynamical transition between localized and delocalized states, a key problem that remains open in the theory of disordered quantum systems. Pastur, one of the founders of the theory of random Schrödinger operators, has taken a central role driving the study of entanglement entropy in discrete random systems with exciting developments relating the asymptotics of the entanglement entropy with the spectral type of the operator, see [15, 29]. Among the main challenges the theory of random systems has faced in the last decade is to find a rigorous description of localization in many-body systems. Here, quantum spin systems has been a particularly active field [22, 17, 16]. These new directions of research complement the developments in long-standing open problems in the theory that have only recently been tackled, as the proof of Anderson localization for Bernoulli Anderson models on the two dimensional lattice by Ding and Smart [12], followed by the more recent work of Li [26] and Li and Zhang [27] on localization in two and three-dimensional lattices, and the proof of Minami estimates in the continuous setting by Dietlein and Elgart [13]. These eigenvalue spacing estimates have been known to hold in the discrete setting since the 90s and are a crucial ingredient in the proofs of Poisson eigenvalue statistics in the region of Anderson localization. The work of Dietlein and Elgart, therefore, opens the way for a better understanding of the localization region in the continuous setting. Eigenvalue statistics are another criteria that is expected to reveal information on the spectral transition conjectured in disordered quantum systems, together with the behavior of eigenfunctions. On this subject, the recent work of Goldsheid and Sodin [18] stands out providing lower bounds on the eigenfunction decay in terms of the Lyapunov exponents. The latter is the subject of a rich literature that is strongly related to quasiperiodic systems, and new findings in this area have the potential to impact also the research done in random systems, like the recent work of Goldsheid [19] and Gorodetski-Kleptsyn [20, 21] on products of non-iid random matrices that could pave the way to study correlated random systems.

The field of quasiperiodic operators has been very active during the recent years and there are many important results and developments. While the phenomena are presumably general, the rigorous results which we now survey are mostly confined to particular models such the Almost Mathieu operator, extended Harper's model and Maryland model.

First, in the quasiperiodic setting the spectral transition between the localized and the delocalized phases occurs as one varies both the strength of the potential and the arithmetic properties of the parameters, and particularly the frequency. In the case of the Almost Mathieu operator, the arithmetic aspects of the transition, conjectured a couple of decades ago, have been recently settled in the works of Jitomirskaya-Liu [24, 25], Liu [28] Avila-You-Zhou [4], and others. Next, Avila, Last, Shamis, and Zhou [3] recently discovered that not only the spectral type, but also the properties of the spectrum such as homogeneity and the Parreau–Widom condition depend sensitively on the arithmetic properties of the parameters: for well-approximated frequencies, these properties fail, and, moreover, the modulus of continuity of the integrated density of states is essentially the worst possible. Another important development, due to Jitomirskaya and Liu [24, 25], is the discovery of a universal hierarchical structure of eigenfunctions in the localization regime, which confirms some predictions in the physics literature, dating back to Azbel's work in 1964 [5]. Moreover, a new phenomenon (not even previously described in the physics literature) of reflective hierarchy of eigenfunctions was discovered.

It would be extremely interesting to know whether any of these phenomena occur for other quasiperiodic operators. This question remains elusive and it is clear that to tackle it a new set of ideas is required, which we believe can be achieved by fostering interaction with different neighbouring fields. As an additional illustration of the productivity of such cross-field interaction, we mention the recent work of Shamis-Sodin [30] on the quantum dynamics for quasiperiodic operators, which uses a combination of single-energy large deviation estimates with harmonic measure estimates from classical analysis to establish upper bounds. Previously such estimates were known only in one dimension, using case-by-case arguments.

3 Presentation Highlights

3.1 Anton Gorodetski: On the spectrum of 1D Schrodinger operators with random noise

The spectrum of a discrete Schrodinger operator with periodic potential is known to be a finite union of intervals. The same is true for the Anderson Model, i.e. for a Schrodinger operator where the potential is defined be a sequence of i.i.d. random variables. The intermediate case of deterministic aperiodic potentials, or one dimensional quasicrystals (Fibonacci Hamiltonian, Sturmian, Almost Mathieu, limit periodic, substitution potentials, etc.), tend to present a Cantor set as a spectrum, even if it is not easy (or even notoriously hard) to prove in many cases.

The talk was concerned with the following question: what happens if one adds some random noise on top of an aperiodic potential, or, more generally, a given ergodic potential? It turns out that in many cases, the randomness wins, both in terms of the spectral type and in terms of the topological structure of the spectrum. More specifically, one can prove Anderson Localization for such models, i.e. one can show that the spectrum must be pure point almost surely. And, under the additional assumption that the phase space of the dynamical systems that defines the background potential is connected, one can show that the almost sure spectrum must be a finite union of intervals, exactly as in the Anderson Model. In particular, the spectrum of a quasiperiodic 1D Schrodinger operator with iid random noise is a finite union of intervals.

The results presented are based on a joint project with V.Kleptsyn, as well as on recent results joint with A.Avila and D.Damanik.

3.2 Zhenghe Zhang: Anderson localization for potentials generated by hyperbolic transformations

This talk presents recent joint work with A. Avila and D. Damanik, showing Anderson localization for Schrödinger operators generated by hyperbolic transformations. Specifically, they consider a topological mixing subshift of finite type with an ergodic measure admitting a bounded distortion property and show that if the Lyapunov exponent has uniform positivity and uniform large deviations on a compact interval, then the operator has Anderson localization on that interval almost surely. For potentials that have a small supremum norm or that are locally constant, they establish a certain uniform large deviations theorem which together with their previous work on positivity of the Lyapunov exponent yields full spectral localization. In particular, these general results can be applied to the doubling map, Arnold's cat map, or a Markov chain.

3.3 Peter Mller : On the return probability of the simple random walk on Galton-Watson trees

The simple random walk on GaltonWatson trees with supercritical offspring distribution, conditioned on non-extinction, was considered. In case the offspring distribution has finite support, an upper bound was shown for the annealed return probability to the root which decays subexponentially in time with exponent 1/3. This exponent is optimal. This result improves the previously known subexponential upper bound with exponent 1/5 by Piau [Ann. Probab. 26, 10161040 (1998)]. For offspring distributions with unbounded support but sufficiently fast decay, this method also yields improved subexponential upper bounds.

3.4 Jacob Shapiro : Classification of 1D Chiral Insulators in the mobility gap regime

Topological insulators are usually studied using a spectral gap condition at the Fermi energy. However, physically it is more interesting to employ the insulator condition via Anderson localization, i.e., forgo a spectral gap and assume the Fermi energy is surrounded by eigenvalues corresponding to localized states. I will describe the problem of topological classification of insulators, in particular in one-dimension, in this Anderson localized regime of insulators.

3.5 Frédéric Klopp : The ground state of a system of interacting fermions in a random field: localization, entanglement entropy

Transport in disordered solids is a phenomenon involving many actors. The motion of a single quantum particle in such a solid is described by a random Hamiltonian. Transport involves many interacting particles, usually, a small fraction of the particles present in the material. One striking phenomenon observed and proved in disordered materials is localization: disorder can prevent transport! While this is quite well understood at the level of a single particle, it is much less clear what happens in the case of many interacting particles. Physicist proposed a number of tools (exponential decay of finite particle density matrices, entanglement entropy, etc) to discriminate between transport and localization. Unfortunately, these quantities are very difficult to control mathematically for "real life" models. We'll present a toy model where one can actually get a control on various of these quantities at least for the ground state of the system. The talk is based on the PhD theses of and joint work with N. Veniaminov and V. Ognov.

3.6 Jake Fillman: The spectrum of the doubling map model is an interval

The talk explains the following phenomenon: no half-line Schrödinger operator or Jacobi matrix generated by the doubling map has any (essential) spectral gaps. It was previously only known that the (essential) spectrum contains an interval, and hence is not nowhere dense, but there was no additional information on the possible existence of gaps. The proof uses Johnson's formulation of gap-labelling: the value of the IDS in the gap coincides with the value of the Schwartzman homomorphism applied to the stable section of the transfer matrix cocycle. This enables one to extract a contradiction from the assumption of a nontrivial gap. The case if singular Jacobi matrices necessitates some preparatory work, as Johnson's gap labelling theorem was previously not available in this case. Such a theorem is established along the way, taking advantage of recent work of Alkorn and Zhang on a deterministic version of Johnson's characterization of the (essential) spectrum. The talk is based on joint work with David Damanik (Rice University), Iris Emilsdottir (Rice University), and Zhenghe Zhang (UC Riverside).

3.7 Xueyin Wang: Recent progress on non-self-adjoint quasi-periodic operators

Non-self-adjoint operators have attracted widespread attention due to their importance in mathematical physics. In particular, the theory of the quasi-periodic Schrdinger operator with complex-valued potential has made many breakthroughs in recent years. The talk reports some recent progress along with some open questions on this topic.

Winding number and density of states are two fundamental physical quantities for non-self-adjoint quasi-periodic Schrdinger operators, which reflect the asymptotic distribution of zeros of the characteristic determinants of the truncated operators under Dirichlet boundary condition, with respect to complexified phase and the energy, respectively. We will prove that the winding number is in fact Avilas acceleration, and it is also closely related to the density of states by a generalized Thouless formula for non-self-adjoint Schrdinger operators and Avilas global theory. The proof is based on the large deviation estimates and the Jensen formula on the annulus.

For non-self-adjoint almost-periodic Schrdinger operators, a criterion is given to guarantee that they have both the same spectrum and the same Lyapunov exponents as the discrete free Laplacian. As a byproduct, it is shown that the Moser-Pschel argument for opening gaps may not be valid for non-self-adjoint operators. We provide a new proof for the spectrum result of Sarnaks model in P. Sarnak: Spectral behavior of quasi periodic potentials, Commun. Math. Phys., 84(3), 377-401, 1982.

We provide a precise formula for the spectrum of the Hatano-Nelson model with strictly ergodic potentials in terms of its Lyapunov exponent. As applications, one clearly observes the real-complex spectrum transition. Moreover, if the Lyapunov exponent is continuous, the spectrum of the Hatano-Nelson model in $l^2(Z)$ can be approximated by the spectrum of its finite-interval truncation with periodic boundary conditions. Both of these results are strikingly different from the Hatano-Nelson model with random potentials, for example, see E. B. Davies: Spectral theory of pesudo-ergodic operators, Commun. Math. Phys., 216(3), 687-704, 2001.

3.8 Matthew Powell: Continuity of the Lyapunov exponent for Gevrey cocycles

Many spectral properties of 1D Schrödinger operators have been linked to the Lyapunov exponent of the corresponding Schrödinger cocycle. In particular, (the regularity of) the integrated density of states, a quantity of particular interest in physics, is related to the regularity of the Lyapunov exponent via a Hilbert-type transform.

The situation for one-frequency quasi-periodic operators with analytic potential is well-understood; indeed, the works of Goldstein-Schlag, Bourgain-Jitomirskaya, and Avila-Jitomirskaya-Sadel yield a more-or-less complete picture in the 1-frequency analytic setting. Neither the multifrequency nor the non-analytic situations are as well understood. The purpose of this talk is twofold: first, we will discuss our recent work establishing continuity (both in cocycle and jointly in cocycle and frequency) of the Lyapunov exponent for non-identically singular analytic cocycles. This extends a result of Bourgain from 2005 for SL(2,C) cocycles. As a corollary of our argument, we also establish a novel weak modulus of continuity for the continuity in frequency.

Secondly, we will discuss ongoing work extending these results to suitable Gevrey classes. It is well-known that discontinuities for the Lyapunov exponent arise in the space of smooth cocycles, but work by Klein and Ge-Wang-You-Zhou indicates that, at least under suitable restrictions, these discontinuities can be avoided. Results for analytic one-frequency cocycles have been known for over a decade, but the multi-frequency results have been limited to either Diophantine frequencies (continuity in cocycle) or SL(2,C) cocycles (joint continuity). We will present ongoing work which extends beyond SL(2,C) cocycles.

We will discuss the main points of our argument, which extends earlier work of Bourgain.

3.9 Xiaowen Zhu: Topological edge spectrum along curved interface

Topological insulators are revolutionary phases of matter that have been intensively studied in the past decades. They are insulating in their bulk but support stable currents along their boundary. This unique property presents potential for technological leaps in the conception of new robust electronic devices, needed for instance in quantum computers.

One of the most important property of TI is the bulk-edge property mentioned above is supposed to be robust w.r.t. deformation of the material. In this talk, we investigate how robust this property is, i.e. under which shape of the material does the property still hold.

More explicitly, we first provide a concise proof that if the boundary of a topological insulator divides the plane in two regions containing arbitrarily large balls, then it acts as a conductor. Conversely, we construct a counterexample to show topological insulators that fit within strips do not need to admit conducting boundary modes. Finally, we show that though there exists counter examples, the topological insulators fitting within strips has a strong tendency to have bulk-edge properties, with most energies. It is worth mentioning that our proof relies on a seemingly paradoxical and under-appreciated property of the bulk indices of topological insulators: they are global quantities that can be locally computed. The talk is based on a joint work with Alexis Drouot.

3.10 Svetlana Jitomirskaya: Dual Lyapunov exponents and robust Ten Martini Problem

We discuss the duality approach to Avila's global theory, developed in Ge-J-You-Zhou, that led to solutions of several outstanding spectral problems, including the almost reducibility conjecture (L. Ge) and the ten martini problem (Cantor spectrum with no condition on irrational frequencies), previously known only for the almost Mathieu operator, for a large class of one-frequency analytic quasiperiodic operators, including nonperturbative analytic neighborhoods of several popular explicit families (Ge-J-You). We then give more detail on the latter proof.

3.11 Long Li: Exact mobility edges for almost periodic CMV matrices

In this talk I will present a proof of explicit mobility edges for the so-called unitary version of the almost Mathieu operators. We show the coexistence of pure point spectrum with exponential decaying eigenfunction and absolutely continuous spectrum.

The proof proceeds in three aspects. The first part is the explicit computation of the Lyapunov exponents of the almost periodic cocycles through Avilas quantization of acceleration and convexity of the complexified Lyapunov exponent. Although our model is not technically quasi-periodic, however, by combing four steps together, we do obtain a quasi-periodic model. As a consequence, the Lyapunov exponent depends on the spectral parameter and displays positive and zero values in its range.

The second part of the proof is a key observation that the associated unitary matrix of the operator (quantum walk) can be placed in a bigger family which are unitarily equivalent to each other by diagonal unitary gauge transforms. The benefit of this is that we can treat the cocycle as SU(1,1) valued when we need results from global theory and reducibility and don not have to deal with the extra phase factor. Moreover, it gives us the freedom to introduce a reflection symmetry by which we can apply Jitomirskayas trigonometric polynomial analysis. Therefore, we give an arithmetic version of localization with explicit decay rate of eigenfunctions provided the Lyapunov exponent is positive.

The third part of the proof is the absolutely continuous spectrum for zero Lyapunov exponent. The proof is a combination of a global-to-local reduction and a previous work by Li-Damanik-Zhou for quasi-periodic CMV matrices.

3.12 Milivoje Luki: Universality limits via canonical systems

It is often expected that the local statistical behavior of eigenvalues of some system depends only on its local properties; for instance, the local distribution of zeros of orthogonal polynomials should depend only on the local properties of the measure of orthogonality. The most commonly studied case is known as bulk universality, where Christoffel-Darboux kernels have a double scaling limit given by the sine kernel. In this talk, I will discuss a new approach which gave the first completely local sufficient condition for bulk universality, and a second new approach which gives necessary and sufficient conditions for universality limits. This work uses the de Branges theory of canonical systems, and applies to other self-adjoint systems with 2x2 transfer matrices such as Schrodinger operators. The talk is based on joint work with Benjamin Eichinger (TU Wien), Brian Simanek (Baylor University), and Harald Woracek (TU Wien).

3.13 Thomas Spencer: Dynamics of the Nonlinear Schrödinger Equation with bounded initial data

We discuss the behavior of the NLS on R or the discrete NLS on Z with bounded initial data. Examples include quasi-periodic and random data. On Z, polynomial bounds are proved for all bounded initial data. In the continuum, local existence is established for real analytic data. We also discuss the long time behavior of a smoothly driven anharmonic oscillator. This is joint work with B. Dodson (Johns Hopkins University) and A. Soffer (Rutgers University).

3.14 Simon Becker: TBG vs TMDs

I review properties of two classes of moire materials, twisted bilayer graphene (TBG) and twisted semiconductors (TMDs) with an emphasis on their mathematical differences. I also discuss results on these models under disorder. In particular, this provides a mathematical account of the recent Physical Reviews Letter by Tarnopolsky–Kruchkov–Vishwanath. The new contributions are a spectral characterization of magic angles, its accurate numerical implementation and an exponential estimate on the squeezing of all bands as the angle decreases. Pseudospectral phenomena due to the non-hermitian nature of operators appearing in the model play a crucial role in the analysis. We then consider small random perturbations of the standard high-symmetry tunneling potentials in the Bistritzer-MacDonald Hamiltonian describing twisted bilayer graphene. Using methods developed by Sjöstrand for studying the spectral asymptotics of non-selfadjoint pseudo-differential operators, we prove that for sufficiently small twisting angles the Hamiltonian will not exhibit a flat band with overwhelming probability, and hence the absence of the so-called magic angles. Moreover, we prove a probabilistic Weyl law for the eigenvalues of the non-selfadjoint tunneling operator, subject to small random perturbations, of the Bistritzer-MacDonald Hamiltonian in the chiral limit. The talk is based on joint work with Maciej Zworski, Izak Oltman, Martin Vogel, and Mengxuan Yang.

3.15 Jun Yin: Non-mean-field random matrices related to quantum chaos and Anderson conjecture

The Quantum Chaos Conjecture has long captivated the scientific community, proposing a crucial spectral phase transition demarcating integrable systems from chaotic systems in quantum mechanics. In integrable systems, eigenvectors typically exhibit localization with local eigenvalue statistics adhering to the Poisson distribution. In contrast, chaotic systems are characterized by delocalized eigenvectors, and their local eigenvalue statistics reflect the Sine kernel distribution reminiscent of the conventional random matrix ensembles GOE/GUE. Similarly, the Anderson conjecture reveals comparable phenomena in the context of disordered systems.

This talk delves into the heart of this phenomenon, presenting a novel approach through the lens of random matrix models. By utilizing these models we aim to provide a clear and intuitive demonstration of the same phenomenon shedding light on the intricacies of these long-standing conjectures.

3.16 Lingfu Zhang: 3D lattice Anderson-Bernoulli localization

I will talk about the lattice Anderson model (i.e., the random Schrödinger operator of Laplacian plus independent identically distributed (i.i.d.) potential), which is widely used to understand the conductivity of materials in condensed matter physics. A key phenomenon is Anderson localization, which was rigorously established in the 1980s, with one remaining question being the Bernoulli potential case. A continuous-space analog of this problem was solved by Bourgain and Kenig [Inventiones 2005]; and more recently, the 2D lattice setting was solved by Ding and Smart [Inventiones 2020]. I will discuss how we further proved the 3D lattice Anderson-Bernoulli localization near spectrum edges, based on the framework developed in these works. I will focus on the extra difficulties in the 3D lattice setting, and explain our main contribution, which is the development of a 3D discrete unique continuation principle. This is joint work with Linjun Li.

3.17 Wei-min Wang: On the dynamics of nonlinear random and quasi-periodic systems

We discuss the time evolution of nonlinear random and quasi-periodic systems on the lattice and elaborate on their similarities. Specifically, we consider the nonlinear Schrödinger equation and the non-linear wave equation on the standard d-dimensional lattice with a random or quasi-periodic potential and discuss, under suitable assumptions, the existence of solutions that are localized in space and quasi-periodic in time as well as long time Anderson localization for the Cauchy problem with general square-summable initial data. The results reported on are joint with Jean Bourgain, Hongzi Cong, Jiansheng Geng, Wencai Liu Liu, Ilya Kachkovskiy, Yunfeng Shi, Yingnan Sun, and Zhifei Zhang.

3.18 Ilya Goldsheid: Products of random Schrödinger matrices depending on a parameter: the strip case

Questions about the properties of products of random matrices depending on a parameter naturally arise in the theory of non-self adjoint random Schrdinger operators on a strip. I review several old and recent results concerning such products in the case of self- and non-self adjoint random Schrödinger operators on a strip and state some open questions.

3.19 Charles Smart: Localization of random band matrices

3.20 Igor Krasovsky: Gap Probability for the Freud ensemble of random matrices: strong and weak confinement regimes

The Freud ensemble of random matrices is the unitary invariant ensemble We corresponding to the weight $\exp(-n|x|\beta)$, $\beta > 0$, on the real line. consider the local behavior of eigenvalues near zero, which exhibits a transition in β. If $\beta \geq 1$, it is described by the standard sine process. Below the critical value $\beta = 1$, it is described by a process depending on the value of β, and we determine the first two terms of the large gap probability in it. This so called weak confinement range $0 < \beta < 1$ corresponds to the Freud weight with the indeterminate moment problem. We also find the multiplicative constant in the asymptotic expansion of the Freud multiple integral for $\beta \geq 1$.

4 Scientific Progress Made

The event was characterized by a lively exchange of ideas, with active discussions taking place not only during the presentations but also throughout the breaks and meal times. Additionally, a dedicated problem session was organized on Tuesday evening, offering participants a focused opportunity to delve into specific challenges.

5 Outcome of the Meeting

The workshop sparked dynamic and interdisciplinary conversations, highlighting the value and possibilities of collaboration across various fields. Leading experts along with junior researchers engaged in in-depth discussions, both during formal presentations and through informal interactions. This gathering provided the opportunity for future collaborations within the realms of ergodic operators, random matrix theory, and statistical mechanics. During the open problems session on Tuesday evening, the audience proposed several challenges for participants to tackle collaboratively. Here are some selected open problems from that session (contributed by David Damanik and Xiaowen Zhu):

- Robustness of the bulk-edge property under random perturbations - The bulk-edge property mentioned above is not only robust w.r.t. deformation of the material, but it is also expected to be robust w.r.t. random perturbations. In [14], the authors introduced the notion of “mobility gap to characterize the random perturbation and proved emergence of edge spectrum for topological insulators with a straight edge and mobility gap. It is expected that similar results will hold in the curved edge case - under our first condition.

- Spectral type and propagation properties of the edge spectrum - The bulk-edge property mentioned above is mathematically characterized by the so-called “emergence of edge spectrum. Now since the edge spectrum emerged, it is natural to ask what is the spectral type of it. Different edge spectral type will leads to different pattern of the edge modes. Most physicists believe the spectrum should be absolutely continuous. However, mathematically this question is not even fully answered in the straight edge case.

- Bulk-edge correspondence under deformation - Apart from emergence of the edge spectrum, there is more delicate structures of the bulk-edge property, that is, the bulk-edge correspondence. It states that the bulk index, an integer determined by the material, equals to the edge index, i.e. the average density of edge modes within a gap. It has been intensively studied for different models under different regime - but with straight edge. It is also natural to ask whether it remains true in the curved edge case.

- Asymmetric transportation - Another special property of the topological insulators is the asymmetric transportation of the edge states, i.e. the edge states induced by the edge spectrum only propagate in one direction along the edge, not the other way around. This effect has been proved in some specific model [6] but remains open in general, especially for curved edge.

- Random potentials without independence - The work [1, 2] establishes Anderson localization for Schrödinger operators with potentials built from non-local sampling over a subshift of finite type. This shows in particular that localization properties are preserved if one introduces local correlations in a Bernoulli-type Anderson model. This work however does not cover cases where the Anderson model one starts out with does not have a finitely supported single-site distribution. On the other hand, our intuitive understanding of random models suggests that increasing randomness should lead to more singular spectral types, and the latter model is more random than the former in a natural sense. Thus, the open problem to address here is to find a mechanism that can establish Anderson localization for non-local sampling over general Anderson-type potentials.

- A more comprehensive understanding of the case of potentials taking finitely many values - There is a general tendency for sampling functions taking finitely many values to produce operators with zero-measure Cantor spectrum and singular continuous spectral measures. For the concrete case of irrational circle rotations as the base dynamics, the zero-measure property for example has been understood for some time [11]. However, generalizing this to minimal translations on higher-dimensional tori appears to be difficult. There is some recent progress, [7], but new ideas will be needed to understand this issue more deeply.

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