# Bootstrap Percolation and its Applications

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## 1 Overview

Bootstrap percolation has its origins in the work of Chalupa, Leath and Reich [?] on disordered magnets. As the concentration of an impurity in a magnetic substance increases, more electrons are unable to interact with their neighbors. At a critical concentration, the strength of the magnet decreases dramatically, and a sharp phase transition is observed.

In fact, bootstrap percolation processes are related to cellular automata, which were defined and studied much earlier, by Ulam [?] and von Neumann [?]. In such processes, sites update their status depending on the behavior of their local neighborhood. Bootstrap percolation is a simple, monotone example: each site is either inactive or active, and once sites become active they stay active forever. Bootstrap percolation is also related to weak saturation in graphs, as introduced by Bollobás [?].

For a detailed survey, see Morris [?]. In what follows, we focus on the topics that received the most attention during this workshop.

### 1.1 Metastability

The first results in the field studied bootstrap percolation on trees and lattices, but the process can be defined on any (potentially infinite or random) graph $G = (V, E)$. In the most basic setting, called $r$-neighbor bootstrap percolation, each vertex is initially active with probability $p$. Subsequently, vertices are activated once (if ever) at least some threshold $r$ of its neighbors become infected. The process percolates if eventually all vertices in $V$ become infected. Results by van Enter [?] and Schonmann [?] showed that on $\mathbb{Z}^d$ the critical density $p_c$ is trivial, either 0 or 1.

One way to obtain a non-trivial phase transition is to study $[n]^d$. In this context, we want to know how $p_c \to 0$ as $n \to \infty$. The first results in this direction are contained in the landmark work of Aizenman and Lebowitz [?], who showed that on $[n]^d$ with $r = 2$, we have $p_c = \Theta(1/\log^{d-1} n)$.

Several ideas in this article have been influential. One such idea is the observation that the dynamics of many bootstrap percolation processes evolve by nucleation, in which case percolating graphs have, roughly speaking, percolating subgraphs of all orders. In the case of $[n]^d$ with $r = 2$, in each time step the size of the largest percolating subgraph can at most (essentially) double. This is very useful when proving lower bounds on $p_c$, as it can suffice to show, in the subcritical regime of the model, that there are no percolating structures in some well-chosen interval (typically of logarithmic size).

Another such idea is that of a critical droplet, that is, the emergence of a percolating subgraph in the supercritical regime that becomes essentially unstoppable, leading to full percolation. In the case $[n]^2$ with $r = 2$, such a droplet corresponds to a percolating rectangle of a critical, logarithmic size. The emergence of such a critical droplet is referred to as metastable behavior.

Cerf and Cirillo [?] and Cerf and Manzo [?] extended the results in [?] to all $r \le d$. In breakthrough work, Holroyd [?] showed that, when $r = d = 2$, we have that $p_c \sim (\pi^2/18)/\log n$. Balogh, Bollobás, Duminil-Copin and Morris [?] later found the sharp threshold for all $r \le d$.

## 1.2 Locality

The bootstrap percolation paradox refers to the fact that numerical estimates for $p_c$, when $d = r = 2$, have been far from the truth. For instance, estimates for the constant $A$ in $p_c \sim A/\log n$, prior to Holroyd's proof that $A = \pi^2/18$, were off by about a factor of 2. The reason for this, roughly speaking, is that the second order term in the expansion for $p_c$ does not become of lower order influence until $n$ is well outside computational range for a straightforward Monte Carlo algorithm. More details about the history and reasons for such discrepencies are given in the recent work of Hartarsky and Teixeira [?, ?].

In [?, ?], the authors resolve the paradox, by showing that 2-neighbor bootstrap percolation on the plane is local, in the following certain sense. Let $\tau$ be the first time (possibly $\tau = \infty$) that the origin is active. Likewise, let $\tau_{\mathrm{loc}}$ be the first time the origin is active in the local model, where, roughly speaking, the activation spreads locally starting for some initially active vertex. In [?], it is shown that

$$\lim_{p\to 0} \mathbb{P}_p \left(1 \le \tau_{\mathrm{loc}}/\tau \le \exp\left[\log^C(1/p)\right]\right) = 1.$$

This result is very useful, since the local model is simpler to analyze. Using this, it is shown that, with probability tending to 1 as $p \to 0$, we have that

$$\frac{\pi^2}{18}\frac{1}{p} - \frac{\lambda}{\sqrt{p}} - \frac{\log^2(1/p)}{\sqrt[3]{p}} < \log \tau < \frac{\pi^2}{18}\frac{1}{p} - \frac{\lambda}{\sqrt{p}} + \frac{\log^2(1/p)}{\sqrt[3]{p}},$$

where $\lambda$ is an explicit constant. This work offers a new perspective on bootstrap percolation (avoiding, in particular, the hierarchy arguments in [?]), and improves on several important works in the field by Aizenman and Lebowitz [?], Holroyd [?], Gravner and Holroyd [?], Gravner, Holroyd and Morris [?], Morris and Hartarsky [?], and others.

## 1.3 Polluted Models

Another way to obtain a non-trivial phase transition for $r$-neighbor bootstrap percolation on $\mathbb{Z}^d$ is to add pollution. In this setting, vertices are initially active with probability $p$ and polluted/deleted with probability $q$. Polluted vertices can never become active, thereby creating a set of obstacles, so to speak, for the spread of activation. Gravner and McDonald [?] initiated the study of such processes. When $\mathbb{Z}^2$ and $r = 2$, the critical scaling is $q/p^2$. Specifically, when $q < cp^2$ the origin is activated eventually with probability tending to 1 as $p \to 0$. On the other hand, when $q > Cp^2$, this probability tends to 0 as $p \to 0$. Gravner and Holroyd [?] and Gravner, Holroyd and Sivakoff [?] have made recent progress in higher dimensions.

## 1.4 Weak Saturation

The study of critical droplets can lead to challenging combinatorial questions. Such structures tend to be extremal, so as to percolate efficiently (e.g., with the minimal number of edges). The emergence of a critical droplet is a rare event when $p$ is close to $p_c$.

Bollobás [?] introduced concepts in combinatorics related to bootstrap percolation. In this setting, we consider the spread of activation on edges rather than vertices. The update rule is governed by some fixed graph $H$. Initially, all edges in some subgraph $G$ of the complete graph $K_n$ are present. We then iteratively add edges if they complete a copy of $H$. If we eventually add in all missing edges, and obtain $K_n$, we say that the process percolates, or that $G$ is weakly $H$-saturated.

Bollobás found (by an edge-switching argument) the minimal number of edges in a weakly $K_r$-saturated graph, for small values of $r$. This was later proved for general $r$ by Lovász [?], Frankl [?], Kalai [?, ?] and Alon [?]. These works have sparked many subsequent works in extremal combinatorics, and it remains an active field of research today.

To obtain a random process, Balogh, Bollobás and Morris [?] introduced a variant of bootstrap percolation, called graph bootstrap percolation. In this process, we study the weak $H$-saturation dynamics, initialized by the Erdős–Rényi graph $G(n, p)$. The classical $r$-neighbor bootstrap percolation dynamics on $G(n, p)$ were also studied around the same time, by Janson and Łuczak, Turova and Vallier [?].

# 2 Mini-Course Lectures

Three mini-course lectures were given early in the week, which aimed to be accessible to all participants. The first introductory lecture by Rob Morris gave a broad overview of the current state of the field, and discussed a number of open problems. The next two lectures by Ivailo Hartarsky discussed in more detail the state-of-the-art in the specific case of 2-neighbor boostrap percolation in $d = 2$ dimensions.

## 2.1 Rob Morris (recorded)

Title: Bootstrap percolation and other automata
Summary: Morris began with the standard $r$-neighbor model. We then turned to the more general $\mathcal{U}$-bootstrap processes introduced by Bollobás, Smith and Uzzell [?]. In this setting, there is a finite collection $\mathcal{U}$ (called the update family) of finite subsets of $\mathbb{Z}^2$. A site $x$ becomes infected if $x + U$ is infected for some $U \in \mathcal{U}$. There are three classes (defined in terms of stable directions) of such processes: supercritical, critical and subcritical.

In [?], it is shown that supercritical processes have polynomial thresholds $p_c = n^{-\Theta(1)}$ and that critical processes have poly-logarithmic thresholds $p_c = (\log n)^{-\Theta(1)}$. They conjectured, and Balister, Bollobás, Przykucki and Smith [?] proved, that subcritical families have thresholds $p_c > 0$, bounded away from 0. More detailed results by Bollobás, Duminil-Copin, Morris and Smith [?] were discussed for critical models. When $\mathcal{U}$ is balanced, in a certain sense, we have that $p_c = (\log n)^{-1/\alpha}$. When $\mathcal{U}$ is unbalanced there is an extra poly-loglog term. Questions remain about when this threshold is sharp; see the recent work by Duminil-Copin and Hartarsky [?].

In a series of works, Balister, Bollobás, Morris and Smith [?, ?, ?] (cf. Hartarsky and Szabó [?] in the subcritical case) investigate $\mathcal{U}$-bootstrap percolation in general $d$ dimensions. As before, there are three classes of families $\mathcal{U}$. There are, however, $d - 1$ types of critical families: for some $2 \le r(\mathcal{U}) \le d$, we have that $p_c = (\log_{(r-1)} n)^{-\Theta(1)}$. Balister, Bollobás, Morris and Smith will show, in forthcoming work [?], that the exponent $\Theta(1)$ is uncomputable in general when $r < d$. When $r = d$, they conjecture that the exponent is computable, which remains a major open problem.

Morris concluded with a number of open problems and remarks, including the following:

- Balogh, Bollobás and Morris [?] found $p_c$ for the hypercube $Q_d$ with $r = 2$. The cases $r \ge 3$ remain completely open.
- Consider zero temperature Glauber dynamics for the Ising model on $\mathbb{Z}^d$, with all sites initially positive $+$, independently with probability $p$. Using ideas from bootstrap percolation, Morris [?] showed that the critical threshold for fixation is $p_c \to 1/2$ as $d \to \infty$. The conjecture that $p_c = 1/2$, for all $d \ge 2$, is a longstanding open problem. It also remains open to show that $p_c < 1$ for $\mathcal{U}$-Ising Glauber dynamics; see Blanquicett [?] for partial results.
- Consider the polluted bootstrap model introduced by Gravner and McDonald [?]. They showed that, when $d = r = 2$ and as $q \to 0$, the critical threshold, at which point an infinite component

of infected sites becomes likely, is $p_c = \Theta(\sqrt{q})$. Morris conjectured [?] that, for $d > r \geq 1$, there is some $q_0(d,r) > 0$ such that $p_c = 0$ for all $0 < q < q_0$. That is, for sufficiently small $q$, arbitrarily small $p$ will be enough to infect an infinite component. In the case that $d = 3$ and $r = 2$, this has been proved by Gravner and Holroyd [?].

- Kinetically constrained spin models were also discussed, which can be thought of as biased random walks on the set of $\mathcal{U}$-bootstrap percolating configurations. See, e.g., the recent works by Martinelli, Morris and Toninelli [?], Hartarsky, Marêché and Toninelli [?], Hartarsky, Martinelli and Toninelli [?], Hartarsky and Marêché [?], and Hartarsky [?].

- Morris says that, given our original motivation from statistical physics, a next frontier worth exploring is that of bootstrap percolation in random geometric settings. Polluted bootstrap percolation is a step in this direction, but there is more to explore. He suggests, for instance, studying bootstrap percolation on random Voronoi tilings.

### 2.2 Ivailo Hartarsky (recorded)

Title: 2-neighbor bootstrap percolation on the plane
Summary: The study of bootstrap percolation started out with the idea that the 2-neighbor model on the square grid with product Bernoulli initial condition would exhibit a nontrivial phase transition, as it does on a regular tree. While this is not the case, the model is far from being trivial. In this mini-course, we will cover the evolution of our quantitative understanding of this trivial phase transition over the years. This will lead us to discuss various techniques, establishing increasingly precise results, many of which have been subsequently adapted to other models and settings. We will revisit the classical works of Aizenman and Lebowitz, and Holroyd, gradually building towards recent and upcoming results in that direction, which still call for generalization.

The first lecture focuses on upper bounds on the infection time (or critical parameter) for a simplified variant of the two-neighbor model. We start by proving Holroyd's sharp threshold bound. We then move on to more precise results and their connection to the bootstrap percolation paradox.

The second lecture is dedicated to lower bounds. We start with the rectangles process and the resulting Aizenman–Lebowitz bound. We then aim to cover the scheme of Holroyd's proof.
References: https://www.normalesup.org/ hartarsky/files/research/talks/2n-biblio.pdf

## 3 Seminar Talks

The following participants were invited to give seminar talks:

### 3.1 Paul Balister (recorded)

Title: Uncomputability in bootstrap percolation
Summary: It is well known that even very simple cellular automata, such as Conway's 'Game of Life' can express extremely complex behaviour, including the ability to emulate a universal Turing Machine. Hence certain aspects of the evolution can be uncomputable, as they can in some cases be equivalent to the Halting problem. Surprisingly, this also holds for generalized bootstrap percolation models in all dimensions $d \geq 2$, despite the fact that such models are highly restricted by being required to be monotone. In particular, we show that when the initial set is given by a random i.i.d. infection with probability $p = p(n)$ in $(\mathbb{Z}/n\mathbb{Z})^d$, even the exponents in the threshold value of $p$ for which percolation occurs can be uncomputable in general.
Joint work with: Béla Bollobás, Robert Morris and Paul Smith [?].

### 3.2 Zsolt Bartha

Title: Critical thresholds in graph bootstrap percolation
Summary: Graph bootstrap percolation is a process introduced by Bollobás in 1968. Fixing a graph $H$, we start from an initial graph $G_0$ and iteratively add edges to it that complete copies of $H$.

When $G_0$ is the Erdős–Rényi random graph $G(n,p)$, there is a critical threshold $p_c$ above which this process will be likely to reach the complete graph $K_n$. We obtain a general asymptotic lower bound for $p_c$, which is sharp in some cases, and matches the upper bound given by Balogh, Bollobás and Morris up to poly-logarithmic factors for so-called balanced graphs $H$. We show that this class of graphs contains $G(k, 1/2)$ with high probability, thus identifying $p_c$ for uniformly random $H$.
Joint work with: Brett Kolesnik and Gal Kronenberg [?, ?].

## 3.3 Alexander Holroyd (recorded)

Title: Polluted bootstrap percolation
Summary: In polluted bootstrap percolation, sites are initially infected with probability $p$ or polluted/removed with probability $q$. Gravner and McDonald [?] initiated the study of such processes, showing that on $\mathbb{Z}^2$ with threshold $r = 2$, the critical scaling is $q/p^2$. In [?], Gravner and Holroyd show that, on $\mathbb{Z}^d$ with $d \geq 3$ and $r = 2$, the density of eventually occupied sites converges to 1 for any $p, q \to 0$. In [?], Gravner, Holroyd and Sivakoff study $\mathbb{Z}^3$ with $r = 3$. In this case, the density of eventually occupied sites converges to 1 if $q \ll p^3/\log^3(1/p)$. On the other hand, the density converges to 0 if $q > Cp^3$ (modified rule) or $q > Cp^2$ (standard rule). The proof involves complicated blocking structures, called "stegosauruses."
Joint work with: Janko Gravner and David Sivakoff [?, ?].

## 3.4 Mihyun Kang

Title: Majority bootstrap percolation on high-dimensional geometric graphs
Summary: Majority bootstrap percolation is a monotone cellular automata that can be thought of as a model of infection spreading in networks. Starting with an initially infected set, new vertices become infected once more than half their neighbours are infected. The average case behaviour of this process was studied on the $n$-dimensional hypercube by Balogh, Bollobás and Morris, who showed that there is a phase transition as the typical density of the initially infected set increases, for small enough densities the spread of infection is typically local, whereas for large enough densities typically the whole graph eventually becomes infected. They showed that the critical window in which this phase transition occurs is bounded away from 1/2, and they gave good bounds on its width.

In this talk we consider the majority bootstrap percolation process on a class of high-dimensional geometric graphs which includes many of the graphs families on which percolation processes are typically considered, such as grids, tori and Hamming graphs, as well as other well-studied families of graphs such as (bipartite) Kneser graphs, including the odd graph and the middle layer graph, and the permutahedron. We show similar quantitative behaviour in terms of the location and width of the critical window for the majority bootstrap percolation process for this class of graphs.
Joint work with: Maurício Collares, Joshua Erde and Anna Geisler [?].

## 3.5 Gal Kronenberg

Title: Independent sets in random subgraphs of the hypercube
Summary: Independent sets in bipartite regular graphs have been studied extensively in combinatorics, probability, computer science and more. The problem of counting independent sets is particularly interesting in the $d$-dimensional hypercube $\{0,1\}^d$, motivated by the lattice gas hardcore model from statistical physics. Independent sets also turn out to be very interesting in the context of random graphs.

The number of independent sets in the hypercube $\{0,1\}^d$ was estimated precisely by Korshunov and Sapozhenko in the 1980s and recently refined by Jenssen and Perkins.

In this talk, we will discuss some results on the number of independent sets in a random subgraph of the hypercube. The results extend to the hardcore model and rely on an analysis of the antiferromagnetic Ising model on the hypercube.
Joint work with: Yinon Spinka [?].

## 3.6 Gábor Pete (recorded)

Title: Volatility in dynamical bootstrap percolation on regular trees
Summary: Consider 2-neighbour bootstrap percolation on the 4-regular infinite tree. The critical density of initial occupation is 1/9, at which point there are already vacant 3-regular subtrees a.s., stopping the bootstrap process from completely occupying the tree. Now let the initial status of every site be resampled according to an independent Poisson process, keeping the density critical. Are there exceptional times in this dynamics when all the vacant 3-regular subtrees get destroyed, hence the entire tree gets bootstrap-occupied?

By a Baire category argument, the existence of exceptional times is equivalent to the set of times at which the root is bootstrap-occupied being everywhere dense a.s. We don't know this, but show semi-volatility: in any time interval $[0, \epsilon)$, starting from the root being bootstrap-vacant, the bootstrapped status of the root changes infinitely many times with probability at least $c\sqrt{\epsilon}$.
Joint work with: Marek Biskup and Àbel Farkas (forthcoming).

## 3.7 David Sivakoff (recorded)

Title: Supercritical neighborhood growth with one-dimensional nucleation
Summary: Supercritical neighborhood growth (or $\mathcal{U}$-bootstrap percolation) rules in two dimensions are classified by the existence of finite configurations that lead to unbounded growth. Bollobás, Smith and Uzzell proved that when the initially occupied set is random with density $p$, the first occupation time of the origin is $p^{-\Theta(1)}$ as $p \to 0$. We take a closer look at supercritical growth rules with +-shaped neighborhoods. These rules exhibit one-dimensional nucleation in the sense that lines begin to grow before forming two-dimensional nuclei. In many cases, we show that the first occupation time is $p^{-\gamma+o(1)}$ for an explicit constant $\gamma$ depending on the rule. In one case, we establish a logarithmic correction to the polynomial passage time due to a growth trajectory that resembles a branching process, while in other cases the first occupation time is of pure polynomial order $p^{-\gamma}$.
Joint work with: Daniel Blanquicett, Janko Gravner and Luke Wilson [?].

## 3.8 Réka Szabó (recorded)

Title: Stability results for random monotone cellular automata
Summary: In a monotone cellular automaton, each site in the $d$-dimensional integer lattice can at each integer time take the values zero or one. The value of a site at a given time is a monotone function of the values of the site and finitely many of its neighbours at the previous time. Toom's stability theorem gives necessary and sufficient conditions for the all one state to be stable under small random perturbations. We review Toom's Peierls argument and extend it to random cellular automata, in which the functions that determine the value at a given space-time point are random and i.i.d. We are especially interested in the case where with positive probability, the identity map is applied, that just copies the value of a site at the previous time. We derive sufficient conditions for the stability of such random cellular automata.
Joint work with: Cristina Toninelli and Jan Swart [?].

## 3.9 Tibor Szabó

Title: Slow subgraph bootstrap percolation
Summary: We study subgraph bootstrap percolation, introduced by Bollobás, where the process is governed by copies of a fixed graph $H$ in the complete graph. We are interesed in the extremal question of the maximum running time, over all possible choices of a starting graph on $n$ vertices. We initiate a systematic study of this parameter, denoted $M_H(n)$, and its dependence on properties of the graph $H$. In a series of works we determine the precise maximum running time for several graph classes. In general, we study necessary and sufficient conditions on $H$ for fast, i.e. sublinear, or linear $H$-bootstrap percolation, and in particular explore the relationship between running time and minimum vertex degree and connectivity. Furthermore we investigate the superlinear regime,

obtain the maximum running time of the process for typical $H$ and discover several graphs exhibiting surprising behaviour.
Joint work with: David Fabian and Patrick Morris [?, ?].

### 3.10 Augusto Teixeira (recorded)

Title: Two-neighbor bootstrap percolation is local
Summary: Metastability thresholds lie at the heart of bootstrap percolation theory. Yet proving precise lower bounds for these quantities is notoriously hard. In this talk we show that for two of the most classical models (two-neighbour and Froböse), the same methodology that is typically used to prove upper bounds can be used to provide lower bounds as well. This is done by linking the models to their local counterparts. As a consequence, we are able to establish the second order term for the infection time of these two models. We will also see how this locality viewpoint can be used to resolve the so-called bootstrap percolation paradox. More precisely, we will present an exact (deterministic) algorithm which exponentially outperforms previous Monte Carlo approaches. We expect this methodology to be applicable to a wider range of models and we finish our talk with a number of open problems.
Joint work with: Ivailo Hartarsky [?].

## 4 Open Problems Sessions

Two open problems sessions were held, on Monday and Tuesday.

Many thanks to Quentin Dubroff, Bob Krueger and Sam Spiro for transcribing the following summary.

### 4.1 Omer Angel

The goal is to understand if certain models of bootstrap percolation are local.

Suppose you are given some sequence of functions $f_n : \{0,1\}^V \to \{0,1\}$, where each function determines an update rule for a process on a vertex transitive graph. The vague question is when do we have $p_c(f_n) \to p_c(f_\infty)$, where $f_\infty$ is some suitably defined limit object. One might require that the sequence $f_n$ is increasing. Perhaps the answer depends on whether the models are subcritical, critical or supercritical.

Question: More concretely, suppose a sequence of graphs $G_n$ tends to a graph $G_\infty$ in the Benjamini–Schramm sense, and pick your favorite bootstrap percolation rule. Is it true that $p_c(G_n) \to p_c(G_\infty)$?

### 4.2 Janko Gravner

Consider bootstrap percolation on the Hamming graph of dimension 2, say on $\mathbb{Z}_+^2$. The process is parametrized by a Young diagram $\mathcal{Z}$, called the zero set. For a given point $p$, count the number of occupied sites in $p$'s column and the number of occupied sites in $p$'s row. If this ordered pair is outside the $\mathcal{Z}$ then we occupy $p$. Let $\gamma(\mathcal{Z})$ be the size of the smallest set that leads to percolation.

When $\mathcal{Z}$ is a rectangle, then $\gamma(\mathcal{Z}) = |\mathcal{Z}|$ is the area of the rectangle.

It is known that $|\mathcal{Z}|/4 \leq \gamma(\mathcal{Z}) = |\mathcal{Z}|$.

Question: Is there an algorithm that computes $\gamma(\mathcal{Z})$ efficiently? Can we approximate $\gamma(\mathcal{Z})$? We suspect that $\gamma(\mathcal{Z}) \geq |\mathcal{Z}|/2$.

See Gravner, Sivakoff and Slivken [?].

### 4.3 Ivailo Hartarsky

Conjecture: For any subcritical $\mathcal{U}$-bootstrap percolation process (i.e., one with $p_c > 0$) we have, for all $p > p_c$, that there exits $c > 0$ such that $\mathbb{P}_p(\tau \geq n) \leq e^{-cn}$.

This is known to hold if $\mathcal{U}$ is "oriented," in the sense that there exists a half space $\mathbb{H}$ through the origin containing all the rules $U \in \mathcal{U}$.

One open case is directed triangle bootstrap percolation.

See Hartarsky [?].

## 4.4 Bob Krueger

The firefighting (single-player) game on a (infinite) graph $G$ is the following: A fire breaks out at a vertex $v$. You may protect (with a "firefighter") $k$ non-burning vertices of $G$ each turn as "unburnable" for the rest of the game (protecting a vertex is equivalent to deleting it from the graph). Between your turns, the fire spreads along the edges of the graph, and a vertex which catches fire burns forever. What is the minimum $k$ needed so that the fire eventually stops spreading?

On $\mathbb{Z}^2$, it is an easy exercise to show that 1 firefighter per turn is not enough, but 2 is.

Conjecture: For the infinite triangular grid, 2 firefighters per turn is not enough (but 3 clearly is).

Conjecture: For the infinite hexagonal grid, 1 firefighter per turn is not enough.

It is known that if you are allowed 1 firefighter per turn, but at on some turn you are given an extra firefighter, then it is possible to contain the fire on the hexagonal grid. There is some relationship between strategies on the triangular grid and strategies on the hexagonal grid.

It is somewhat constraining to allow the same number of firefighters on every turn. You could instead have a (deterministic or random) sequence that tells you how many firefighters you can use. There are many natural variations.

See Finbow and MacGillivray [?].

## 4.5 James Martin

Close sites on $\mathbb{Z}^d$ independently with probability $p$. Consider a two-player game on this site-percolated board, where players alternate moving the location of a token in some specified set of directions $D$ (to a site that is not closed), never repeating a location. A player loses when they can no longer make a move. If a play goes on forever, then the game is a draw. What is the probability of a draw?

When $p$ is so large that there are no infinite components, the game is forced to end, so there is no draw. When $p$ is sufficiently small, is there some positive probability of a draw?

Consider the case $D = \{e_1, \dots, e_d\}$, the standard basis (in the positive directions). For $d = 2$, it is known that the probability of a draw is 0 for all $p$. Label each site with respect to whether the first or second player wins when the token starts at that site. If we know the labels of the diagonal $x + y = c$, then we can determine the labels of the diagonal $x + y = c - 1$. This looks like a 1-dimensional cellular automata, which has a stationary distribution.

Conjecture: For $d \geq 3$ and $D = \{e_1, \dots, e_d\}$, there is a positive probability of a draw, for sufficiently small $p$.

We can prove it for $D = \{e_1 \pm e_i : i \geq 2\}$ and $d \geq 3$. In this case, the game in dimension $d$ is related to a hard-core model (and the uniqueness of the stationary measure) in dimension $d - 1$.

Conjecture: For $D = \{\pm e_i : i \in [d]\}$, the probability of a draw is 0 for $d = 2$, but there is a positive probability of a draw for $d = 3$ and sufficiently small $p$.

The problem is somewhat related to bootstrap percolation (the Froböse model) because enough closed sites "force" other sites to be closed, as a player would never move to the site if it is a losing position.

How much advantage does one player have over the other? The probability of the first player (in dimension 2) is related to a Markov chain computation on a hard-core model. How does closing only even sites on $\mathbb{Z}^d$ shift the advantage?

Consider playing this game on finite graphs (without the site percolation, but, as before, the token may never repeat a site). It is an exercise that the first player wins if and only if the token starts at a vertex which is contained in every maximal matching. To transfer this to infinite volume, we ask: Are the vertices in every maximal matching sensitive to boundary conditions?

See Basu, Holroyd, Martin, and Wästlund [?].

## 4.6 Gábor Pete

Consider bootstrap percolation on Cayley graphs.

A group is amenable if there exists (equivalently, for all) Cayley graph with $F_1 \subseteq F_2 \subseteq \cdots \subseteq V(G)$ such that $\bigcup_{n=1}^{\infty} F_n = V(G)$, $|F_n| < \infty$ for all $n \geq 1$, and $|\partial F_n|/|F_n| \to 0$, where $\partial F_n$ is the set of vertices in $F_n$ with a neighbor outside of $F_n$.

For example, $\mathbb{Z}^d$ is amenable. The Diestel–Leader graph $\mathrm{DL}(2,2)$, a Cayley graph of the lamplighter group, is also amenable.

Question: Prove or disprove: A group is amenable if and only if for every generating set $S$ and for every $k$-rule bootstrap percolation, $p_c(\mathrm{Cay}(\Gamma, S), k) \in \{0, 1\}$.

For $\mathbb{Z}^d$ (amenable) and any group containing a free group with two generators (very non-amenable), the result holds. But for $\mathrm{DL}(2,2)$, it is unknown. The Heisenberg group may also be good to consider.

See Balogh, Peres and Pete [?].

## 4.7 Sam Spiro

The zero forcing process for a graph $G$ starts with some initial set of activated vertices $B_0$. Iteratively, if there exists $v \in B_i$ such that there exists a unique neighbor $u \in N(v)$ which is not in $B_i$, then $u$ gets added to $B_{i+1}$ together with all previous vertices of $B_i$. We write $B_0 \in \mathrm{ZFS}(G)$ if $B_\infty = V(G)$. Note that this property is monotone in that if $B \in \mathrm{ZFS}(G)$ then so is any superset of $B$.

There is a lot of literature studying deterministic $B_0$, but the case when we start with a $p$-random set $B_p$ has only be studied very recently. There are many questions to explore, the main one being the following.

Conjecture: If $G$ is an $n$-vertex graph and $p \in [0,1]$, then $\mathbb{P}[B_p \in \mathrm{ZFS}(G)] \leq \mathbb{P}[B_p \in \mathrm{ZFS}(P_n)]$.

That is, the path is the easiest graph to completely activate with a $p$-random set of vertices. This is known to hold if $G$ has a Hamiltonian path, via a simple coupling argument (which fails for general graphs), and is also known to hold for trees of sufficiently large order.

See Curtis, Gan, Haddock, Lawrence and Spiro [?].

## 4.8 Tibor Szabó

Given two graph $H$ and $G_0$, the $H$-bootstrap percolation process starting with $G_0$ involves iteratively setting $G_i$ to consist of $G_{i-1}$ together with any edges that produce a copy of $H$. We let $\langle G_0 \rangle$ be the graph obtained at the end of the process. We define $M_H(n)$ to be the maximum time it takes for the $H$-bootstrap percolation process on some $n$-vertex graph $G_0$ with $\langle G_0 \rangle = K_n$ to terminate.

Question: Is $M_{K_5}(n) = o(n^2)$?

Note that we have a lower bound $n^{2-o(1)}$ and it is known that for larger cliques $M_{K_r}(n) = \Theta(n^2)$.

Question: Is $M_T(n) \leq e(T)$? for all $n$ sufficiently large.

This would be tight for the star. Currently the best known bound is $O(e(T)^2)$. It is theoretically possible that one might be able to prove a bound of the form $O(\Delta(T))$.

Conjecture: If $M_H(n) = o(n)$ then $M_H(n) = \Theta(\log n)$ or $M_H(n) = O(1)$.

Note that cycles and trees show that either of these behaviors can happen.

Conjecture: $H$ having tree-width 2 implies $M_H(n) = O(n)$.

Question: Are there $H_1$ and $H_2$ such that $M_{H_1 \cup H_2}(N) = \omega(M_{H_1}(n) + M_{H_2}(n))$?

See Matzke [?], Bollobás, Przykucki, Riordan and Sahasrabudhe [?] and Balogh, Kronenberg, Pokrovskiy and Szabó [?].

## 4.9 Maksim Zhukovskii

Consider graph bootstrap percolation. The weak saturation number $\mathrm{wsat}(G, F)$ is the minimum number of edges in a spanning subgraph $H$ of $G$ which percolates to $G$. Recall that, in this process, we start with the edges of $H$. Other edges are added iteratively if they complete a copy of $F$.

For example, $\text{wsat}(K_n, K_s) = \binom{n}{2} - \binom{n-s+2}{2}$. The upper bound is constructive: Take an $(s-2)$-clique in $K_n$ and all edges incident to the clique. There are other constructions. In general, $\text{wsat}(K_n, F)$ is linear in $n$. That is, $\text{wsat}(K_n, F) = (c_F + o(1))n$.

Conjecture: For all $F$ and constant $p$, $\text{wsat}(G(n,p), F) = \text{wsat}(K_n, F)$ with high probability.

We know some $F$ for which this is true, but we do not know $\text{wsat}(K_n, F)$. In particular, for unbalanced complete bipartite graphs, though it is known up to an additive constant by Kalinicheko and Zhukovskii [?].

There exists some $p_s$ such that if $p \gg p_s$ then $\text{wsat}(G(n,p), F) = \text{wsat}(K_n, K_s)$ with high probability, and if $p \ll p_s$ then they are not equal. See Bidgoli, Mohammadian, Tayfeh-Rezaie and Zhukovskii [?].

Question: Find $p_s$.

For this question, we know that $n^{-g(s)} < p_s < n^{-f(s)}$. For triangles, Peled and Zhukovskii (unpublished) have shown that $p_3 = n^{-1/3+o(1)}$.

## 5 Simulations and Applications

A session on computer simulations and applications was held on Thursday, after lunch. A number of people presented simulations and discussed related open problems. Some highlights include:

- Joan Adler presented on applications and kindly provided simulations to post on the external workshop webpage (https://warwick.ac.uk/fac/sci/statistics/staff/academic-research/kolesnik/birs_bp/sims) and a link to AViz software (https://github.com/simphony/AViz).

  See Adler [?], Adler and Lev [?] and Adler, Elfenbaum and Sharir [?].

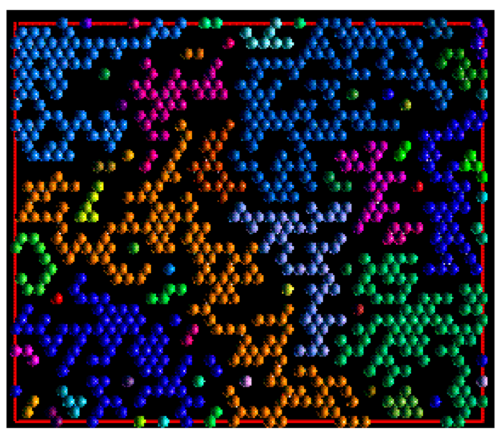

Figure 1: Simulation by Joan Adler and Uri Lev, showing bootstrap percolation on the square grid.

- Not much work has been done on bootstrap rules which are not monotone. As an encouragement to the participants to venture into this new territory, Janko Gravner presented a simulation of a cellular automaton in which the neighborhood of a point is a cross of radius 2 and each unoccupied point requires exactly 2 horizontal or exactly 2 vertical neighbors to become permanently occupied. Started from an initial configuration of low density $p$ of occupied points, the dynamics creates a maze of occupied lines, through which the dynamics tries to occupy more territory. The conjecture is that, as $p \to 0$, the final configuration occupies a limiting portion of space strictly between 0 and 1.

- David Sivakoff presented simulations of the cyclic cellular automaton model on infinite trees, studied in joint work with Janko Gravner and Hanbaek Lyu [?]. In this model, each vertex has a state in $\{0, , \kappa - 1\}$. A vertex in state $k$ switches to state $k+1 \bmod \kappa$ if it has at least one neighbor in state $k+1 \bmod \kappa$.

Figure 2: Simulation by Janko Gravner of the exactly 2 dynamics started from a small $p$, shown at an intermediate time.

A conjecture about this model is that for trees with sufficient growth (branching number $> 1$, say), if the initial configuration is a uniform product measure on $\{0, , \kappa - 1\}$, then, for any $\kappa > 2$, with probability 1 the root changes state infinitely often.

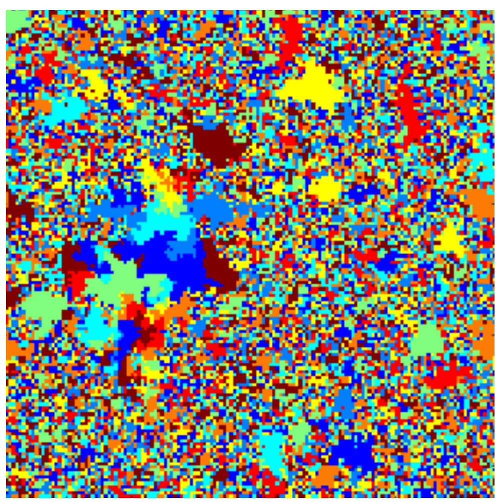

Figure 3: Simulation by David Sivakoff, showing the cyclic cellular automaton model random minimal spanning tree of an $N \times N$ torus with $\kappa = 8$.

## 6 Lightning Talks

On Friday, before checkout, we held a series of three 20-minute lightning talks. The speakers were chosen through an open call to all (in-person and online) participants, with priority given to early career researchers.

### 6.1 Sahar Diskin (online)

Sahar Diskin presented joint work with Ilay Hoshen and Maksim Zhukovskii [?] on the weak and strong saturation numbers of the random graph. In particular, the following conjecture was stated, regarding a possible "jump" in the saturation number: Fix $p \in (0,1)$. Then $\mathrm{sat}(G(n,p),F) = \Theta(n \log n)$ if every edge of $F$ belongs to a triangle in $F$, and $\mathrm{sat}(G(n,p),F) = O(n)$ otherwise.

### 6.2 Bob Krueger

Bob Krueger discussed joint work Igor Araujo, Bryce Frederickson, Bernard Lidický, Tyrrell McAllister, Florian Pfender, Sam Spiro and Eric Stucky [?] on a certain triangle percolation process on the grid, with connections to bootstrap percolation. In this process, we start with some subset $X$ of points in $\mathbb{Z}^2$. Points are iteratively added to $X$ as follows: If there is a triangle with vertices in $\mathbb{Z}^2$ that contains exactly three points in $X$ and exactly four points in $\mathbb{Z}^2$ then we add this additional point to $X$.

### 6.3 Maksim Zhukovskii

Maksim Zhukovskii discussed joint work with Nikolai Terekhov [?] on the limitations of Kalai's linear algebraic method for bounding weak saturation numbers $\mathrm{wsat}(G,F)$. Specifically, $\mathrm{rkwsat}(G,F) \leq \mathrm{wsat}(G,F)$, the latter of which being the maximum rank of a matroid on $E(G)$ that is, in a certain sense, weakly $F$-saturated. In many examples, $\mathrm{wsat}(G,F) = \mathrm{rkwsat}(G,F)$. However, in this work, it is shown that $\mathrm{rkwsat}(K_n,F) < (1-\epsilon)\mathrm{wsat}(K_n,F)$ for infinitely many connected graphs $F$. Alon proved that $\mathrm{wsat}(K_n,F) = c_F n + o(n)$. Tuza conjectured $\mathrm{wsat}(K_n,F) = c_F n + O(1)$. In this work, it is shown that $\mathrm{rkwsat}(K_n,F) = c_F n + O(1)$.

## 7 Panel Discussion

Karen Gunderson organized and moderated a group activity on Thursday, after dinner. The goal of this event was to give useful advice to early career researchers, and those who advise them, about job hunting in the areas represented by this workshop.

The following workshop participants acted as panelists, answering questions and giving advice during the discussion: Omer Angel, Mihyun Kang, Gal Kronenberg, Bernard Lidický and James Martin. Many thanks to Ayush Kumar, who also joined the event virtually as a panelist, specifically to discuss EDI-related topics. Ayush Kumar is Associate Dean and Equity, Diversity, and Inclusion Lead for the Faculty of Science at the University of Manitoba.

We had a number of topics and questions planned in advance, such as: What are the postdoc opportunities in your department/network? How do they work? What are some of the relevant trends in faculty positions? How best to strengthen research statements? Which additional initiatives or activities are most valued on a CV? How best to give meaningful EDI (or DEI) statements? Advice on recruiting a diverse pool of trainees/applicants? How to avoid biased language or themes in reference letters?

The event was well attended. A productive group conversation unfolded at the event, which contributed significantly to a good sense of community throughout the workshop. Participants brought up a wide variety of topics, and the panelists were together able to give careful and meaningful answers.

A document with postdoc and job resources has been posted at the external workshop webpage: https://warwick.ac.uk/fac/sci/statistics/staff/academic-research/kolesnik/birs_bp/birs_bp_resources.pdf.

## 8 Hikes

On Monday, with good weather, many participants hiked up along Tunnel Mountain Trail after lunch, before the afternoon seminars. On Wednesday, a group bus trip to Lake Louise was organized. Many participants hiked across frozen Lake Louise and beyond. Others went elsewhere on self-organized excursions.

# 9 Scientific Progress

Progress was made on several fronts. Some highlights include:

- Ivailo Hartarsky and Augusto Teixeira [?] posted their preprint on the locality of bootstrap percolation (resolving the bootstrap percolation paradox) on the arXiv just a few days before the start of the workshop. This created considerable buzz, and was a great way to start the workshop.
- Janko Gravner, Ander Holroyd and David Sivakoff made progress on a project related to polluted bootstrap percolation in two dimensions.
- Janko Gravner, Ivailo Hartarsky, Ander Holroyd, David Sivakoff and Réka Szabó began a collaboration related to polluted bootstrap percolation in dimensions 3 and higher, making progress on open problems discussed in the talks by Ander Holroyd and Rob Morris. This workshop was successful in connecting these two groups of frequent collaborators, being the first time that Ivailo Hartarsky and Réka Szabó had met in-person with Ander Holroyd and David Sivakoff.
- Mihyun Kang reports that this workshop inspired her to work intensely on several projects, along with various collaborators:
  - with Christoph Koch and Tamás Makai [?], studying bootstrap percolation on the binomial random $k$-uniform hypergraph;
  - with Michael Missethan, and Dominik Schmid [?], studying bootstrap percolation on the high-dimensional Hamming graph;
  - with Maurício Collares, Joshua Erde and Anna Geisler [?], studying majority bootstrap percolation on high-dimensional geometric graphs.
- Mihyun Kang also reports that she will begin working this fall with a new graduate student on a project related to bootstrap percolation in high-dimensional product graphs, with the support of the Austrian Science Fund (FWF).
- Inspired by Gal Kronenberg's seminar on her joint work with Yinon Spinka [?], Shirshendu Ganguly began working on, and recently completed, a project with his students Mriganka Chowdhury and Vilas Winstein [?] on independent sets in random subgraphs of the hypercube.

  This work focuses on the cases $p \in [2/3, 1)$. Specifically, it is shown that the (centered and scaled) partition function is asymptotically Normal for all $p > 2/3$, and the sum of two independent log-normals at $p = 2/3$. The point $2/3$ is a transition point in a certain non-uniform birthday problem.

  The case $p < 2/3$ will be analyzed in forthcoming work.
- Daniel Ahlberg and Christopher Hoffman made progress on questions related to the coalescence of geodesics in first-passage percolation.
- Omer Angel, Janko Gravner and Brett Kolesnik discussed ideas for a future project related to bootstrap percolation in certain random planar geometries.

# 10 Participants

## 10.1 In-Person

There were 42 in-person participants: Daniel Ahlberg, Caio Alves, Omer Angel, Igor Araujo, Paul Balister, József Balogh, Zsolt Bartha, Marcelo Campos, Daniel De La Riva Massaad, Dingding Dong, Quentin Dubroff, Shirshendu Ganguly, Ramón Iván García Alvarez, Janko Gravner, Karen

Gunderson, Ivailo Hartarsky, Jaka Hedžet, Christopher Hoffman, Alexander Holroyd, Mihyun Kang, Brett Kolesnik, Gal Kronenberg, Bob Krueger, Imre Leader, Bernard Lidický, James Martin, Letícia Mattos, Yago Moreno, Yuval Peled, Gábor Pete, Leonardo Rolla, Mihalis Sarantis, David Sivakoff, Erik Slivken, Sam Spiro, Réka Szabó, Tibor Szabó, Augusto Teixeira, Mark Walters, Alexandra Wesolek, Michael Wigal and Maksim Zhukovskii.

### 10.2 Online

There were 51 online participants: Joan Adler, Michael Aizenman, Yago Moreno Alonso, Daniel Blanquicett, Elisabetta Candellero, Altar Çiçeksiz, Sahar Diskin, Damiano De Gaspari, Nils Detering, Alberto Espuny Díaz, Joshua Erde, Nikolaos Fountoulakis, Anna Geisler, Benjamin Gunby-Mann, Lianna Hambardzumyan, Annika Heckel, Cecilia Holmgren, Nina Kamčev, Michael Krivelevich, Emilio Leonardi, Lyuben Lichev, Anita Liebenau, Tomasz Łuczak, Tamas Makai, Laure Marêché, Carlos Martinez, Christian Maura, Marcus Michelen, Dieter Mitsche, Adva Mond, Patrick Morris, Rob Morris, Richard Mycroft, Bhargav Narayanan, Sam Olesker-Taylor, Alexey Pokrovskiy, Daniel Reichman, Dominik Schmid, Roberto Schonmann, Nir Schreiber, Alex Scott, Alexander Scruton, Victor Souza, Alexandre Stauffer, Cristina Toninelli, Fabio Toninelli, Giovanni Luca Torrisi, Andrew Treglown, Daniel Valesin and Aernout van Enter.

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