Perspectives on Markov numbers

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1 Overview of the Field

A *Markov number* is any positive integer in a triple $(x, y, z) \in \mathbb{N}^3$ solving *the Markov Diophantine equation*

$$x^2 + y^2 + z^2 = 3xyz \ .$$

Markov numbers provide a dizzying array of fascinating connections between several active research areas. In Markov's context, these integers appear as the minima of primitive integral indefinite binary quadratic forms. Markov's beautiful theorem from the start of the 20^{th} century provides the first potentially surprising connection, to continued fractions and Diophantine approximation: Markov numbers index the 'most irrational' real numbers. In the mid-20^{th} century, Gorshkov and Cohn separately realized a deep connection to hyperbolic geometry: Markov numbers correspond to traces of simple closed geodesics on the 'equianharmonic torus', a hyperbolic surface covering the modular curve with index 12, with holonomy the commutator subgroup of $PSL(2, \mathbb{Z})$. Later, combinatorial techniques were developed towards the study of Markov numbers, using cluster algebras and 'snake graphs', deepening geometric perspectives on the continued fractions arising from Markov numbers. We consider the appearance of Markov numbers from three perspectives.

1. Hyperbolic Geometry Perspective

Markov numbers correspond to the lengths of simple closed geodesics on the modular torus (a once-punctured hyperbolic torus). This geometric viewpoint, initiated by Cohn and expanded by Series, links Markov numbers to extremal Diophantine approximation problems. Recently, McShane provided a geometric proof of Aigner's conjectures, concerning an ordering on $\mathbb{Q}$ induced by the Markov numbers, using variations in geodesic lengths along moduli space rays [4]. Gaster extended and proved refined versions of these conjectures (originally by Lee-Li-Rabideau-Schiffler), showing monotonicity of Markov numbers along generalized slope rays [6]. Geometric proofs of Markov's theorem now exist using immersed curves and $\mathrm{SL}(2, \mathbb{Z})$ representations (Ian Agol, 2025). Markov numbers also emerge via triangle group actions and quiver mutations, as well as from traces in Fricke's identity [3].

2. Number Theory Perspective

From the classical Diophantine perspective, Markov numbers were first studied by Andrey Markov in relation to minima of indefinite binary quadratic forms. Several major themes continue to drive number-theoretic research. The long-standing *Unicity Conjecture* posits that each Markov number appears uniquely as the largest entry in a Markov triple. The *Strong Approximation Conjecture* of Bourgain-Gamburd-Sarnak [10] asserts that Markov triples mod p form a single orbit for almost all primes. This conjecture is now nearly settled by the combination of the pioneering work of Bourgain-Gamburd-Sarnak [10], together with recent advances of Chen [11] and Eddy-Fuchs-Litman-Martin-Tripeny [15]: Relying on Bourgain-Gamburd-Sarnak, Chen proved the conjecture for all but finitely many p, and Eddy-Fuchs-Litman-Martin-Tripeny sharpened and quantifed these works to show connectivity for $p > 10^{393}$. Martin recently gave a new proof of a special case of Chen's result, contributing to further structural results about divisibility and connectivity in Markoff graphs mod p.

3. Cluster Algebra Perspective

Markov numbers naturally arise in the cluster algebra associated with the once-punctured torus. The recursive mutation structure of cluster variables mirrors the generation of Markov triples. Evaluating cluster variables at 1 recovers the Markov numbers. *Snake graphs* offer a combinatorial model: each Markov number corresponds to a perfect matching count in a planar graph. Using this model, Lee-Li-Rabideau-Schiffler [9] proved Aigner's ordering conjectures and conjectured several refinements. Kaufman-Greenberg-Wienhard [16] recently developed a non-commutative version of the Markov equation via non-commutative cluster algebras, generalizing Markov numbers to new algebraic settings.

2 Recent Developments and Open Problems

Participants in the *Perspectives on Markov numbers* workshop gathered to share their viewpoints on the Markov numbers, to discuss recent developments, and to offer avenues for future research. The workshop began with a trio of introductory talks concerning each of the above viewpoints (delivered by Christopher-Lloyd Simon for hyperbolic geometry, Colby Brown for number theory, and Ryan Schroeder for cluster algebras), and followed with a series of research-level talks, delivered by mathematicians at various different career stages. Participants were encouraged to share ideas for exploration, and time was set aside for participants to work together, share research questions, and offer suggestions for paths forward. Research directions discussed coalesced around the main viewpoints discussed above, which we now describe in more detail.

2.1 Markov Numbers in Hyperbolic Geometry

2.1.1 Markov's Theorem and Geodesics on the Torus

A classical result of Markov is that the *worst approximable* real numbers (those with the largest Diophantine approximation constants) are characterized by Markov triples. Markov showed that there is a discrete set of approximation constants ν_i (Lagrange spectrum values, see Subsection 2.2.2) decreasing to $1/3$, such that if a real number θ has approximation constant $\nu(\theta) > 1/3$, then $\nu(\theta) = \nu_i$ for some i [14]. The corresponding θ are called *Markov irrationals*, and indeed $\nu(\theta) > 1/3$ if and only if θ corresponds to a Markov triple.

In geometric terms, this links Diophantine approximation to simple closed geodesics on the *modular torus* (once-punctured torus corresponding to the commutator subgroup of $PSL(2, \mathbb{Z})$). This hyperbolic interpretation was pioneered by Harvey Cohn, who in the 1950s–1970s developed a viewpoint tying Markov forms to geodesics on the modular surface [7]. Notably, Cohn (1971) showed that every Markov binary form corresponds to a closed geodesic on a punctured torus. Caroline Series further gave a topological proof of Markov's theorem by studying how simple geodesics avoid a certain cusp neighborhood on the modular torus. In particular, Series showed that geodesics corresponding to Markov irrationals are exactly those that spiral into the cusp in a particular way (eventually tracing a simple closed geodesic).

This beautiful picture shows that the worst approximable real numbers are in correspondence with simple closed geodesics on a hyperbolic torus, or equivalently from certain conjugacy classes in $SL(2, \mathbb{Z})$. Highlights from the workshop that explored new aspects of Markov's theorem included Boris Springborn's talk *The worst approximable rational numbers* and Ian Agol's talk *A new proof of the Markov theorem*.

2.1.2 Fricke's Identity

The *Fricke trace identity* provides an algebraic handle on this geometric picture. For any two 2×2 matrices $X, Y \in SL(2, R)$, for any ring R, Fricke's identity gives a cubic relation among the traces $a = \mathrm{tr}(X)$, $b = \mathrm{tr}(Y)$, $c = \mathrm{tr}(XY)$ and $d = \mathrm{tr}([X, Y])$ (the commutator).

In the special case of interest (namely, that arising for a hyperbolic punctured torus), it is natural to consider X, Y such that the commutator is $-I$ (a full rotation), which yields $d = -2$. In that case Fricke's identity reduces to $a^2 + b^2 + c^2 = abc$, or equivalently (a, b, c) (suitably normalized) satisfies the Markov equation. Thus Markov triples can be interpreted as (one-third of the) traces of certain conjugacy classes in $SL(2, \mathbb{Z})$. A novel perspective on Fricke's identity and generalizations was offered at the workshop by M. de Courcy-Ireland, who gave a new proof of Fricke's identity using the spin representation in a 4-dimensional orthogonal group.

2.1.3 Hyperbolic geometry proof of Aigner's Conjectures

Martin Aigner's 2013 monograph *Markov's Theorem and 100 Years of the Uniqueness Conjecture* offered several conjectures concerning Markov numbers that sidestep the thorny difficulty of Unicity – they are neither implied by, nor imply, Unicity. Three of these conjectures concern the relative ordering of Markov numbers. In particular, they predict how Markov numbers grow along certain branches of the Markov tree when projected to the plane. These are often phrased as the *fixed numerator conjecture*, *fixed denominator conjecture*, and *fixed sum conjecture* (referring to lines of slope 0, ∞, or -1 with respect to a natural description of the Markov numbers via coprime integers). Recent breakthroughs have resolved all three of Aigner's conjectures [1, 9] (see Subsection 2.3.2 for more details on this approach).

Greg McShane provided a new proof of Aigner's conjectures by exploiting the hyperbolic geometry of the modular torus and the convexity of a certain norm on its homology [4]. The classical correspondence mentioned above (due to Gorshkov and Cohn) associates each rational $p/q \in [0, 1]$ with a simple closed geodesic on the modular torus $\mathcal{X}$, in such a way that the trace of the associated deck transformation in $\mathrm{SL}(2, \mathbb{R})$ is three times the Markov number $m_{p/q}$. Because the trace of a hyperbolic element is related to the length ℓ of the corresponding geodesic by $2\cosh(\ell/2) = |\,\mathrm{tr}\,|$, the Markov number $m_{p/q}$ can be viewed as a monotonic function of the geodesic length $\ell_{p/q} = \ell(\gamma_{p/q})$: indeed, $m_{p/q} < m_{r/s}$ if and only if $\ell_{p/q} < \ell_{r/s}$ on $\mathcal{X}$. McShane's key insight was to leverage this geometric ordering via the stable norm on homology. The stable norm $\|\cdot\|_s$ on $H_1(\mathcal{X}, \mathbb{R}) \cong \mathbb{R}^2$ is defined so that for any primitive integer class $(q, p) \in H_1$ (which corresponds to a curve of slope p/q), $\|(q, p)\|_s$ equals the hyperbolic length of the unique geodesic in that class. This norm is induced by the hyperbolic length functional and has a remarkable property: its unit ball in $H_1(\mathcal{X}, \mathbb{R})$ is a strictly convex curve.

Using the convexity of this unit ball, McShane gave a streamlined proof of all three monotonicity conjectures of Aigner. The idea can be visualized as follows: consider two homology classes (q, p) and $(q + i, p)$ (horizontal move, fixed p) on the lattice $H_1(\mathcal{X}, \mathbb{Z})$. Because the stable norm ball is strictly convex, moving in a straight line in the (q, p)-plane causes the norm to increase in a predictable way. In particular, $||(q + i, p)||s > ||(q, p)||s$, implying that the geodesic for $(q + i, p)$ is strictly longer than that for (q, p). By the monotonic relation between length and corresponding Markov number, this yields $m_{p/(q+i)} > m_{p/q}$, exactly the Fixed Numerator property. Similar reasoning along vertical lines (fixed q, increasing p) and diagonal lines (increasing q while decreasing p to keep $p + q$ constant) establishes the Fixed Denominator and Fixed Sum conjectures, respectively. In each case, the convexity of the length norm forces a strict inequality in geodesic lengths, hence in Markov numbers. This geometric argument treats all cases in a unified manner and avoids the heavy combinatorial machinery of earlier proofs. McShane's approach provides a more conceptual understanding: the monotonicity is a direct consequence of the convex geometry of closed geodesics on the punctured torus.

Jonah Gaster built on McShane's geometric approach to delve deeper into the Markov ordering. The main result is a complete characterization of which slopes yield a monotonic Markov ordering, confirming and sharpening conjectures by Lee–Li–Rabideau–Schiffler that refined Aigner's predictions. Gaster's approach likewise uses the stable norm $||\cdot||s$ on $H_1(\mathcal{X}, \mathbb{R})$. The unit sphere B of this norm is a convex closed curve which is mostly smooth except at certain primitive lattice points. Gaster computed the slope angles at these corner points explicitly. In practical terms, he found two critical slope values — call them $\sigma_- \approx -1.414$ and $\sigma_+ \approx -1.143$ in slope (i.e. $\Delta p/\Delta q$) — which serve as boundary slopes separating different monotonic regimes. Gaster's computation confirmed specific slope bounds that had been conjectured via cluster-algebra techniques (Lee–Li–Rabideau–Schiffler had predicted monotonicity would fail in a certain window of slopes below $-3/2$).

The Markov ordering of the rationals was explored at the workshop both in Schiffler's presentation *Monotonicity of Markov numbers via perfect matchings of snake graphs*, and in a discussion portion of the problem session led by Gaster.

2.1.4 Hyperbolic Group Actions and Markov Numbers.

Another powerful viewpoint comes from *Kleinian and Fuchsian groups*. For instance, Jørgensen and Gorshkov studied discrete groups related to Markov triples, and Cohn's work can be reinterpreted in terms of *Fricke groups* (subgroups of $SL(2, \mathbb{Z})$ generated by half-turns). In recent work, Anna Felikson and Pavel Tumarkin have considered the geometry of groups generated by three symmetries (half-turn rotations) on the hyperbolic plane and found a direct connection to Markov triples.

In fact, the group generated by three 180° rotations about points arranged in a certain pattern can produce Markov numbers through its limit set or orbit geometry. In Felikson's presentation *Groups generated by three symmetries on the hyperbolic plane* at the workshop, she showed that these configurations correspond to *rank-3 quiver mutations* (related to cluster algebras) and that geometric features of these triangle groups explain known properties of Markov numbers. This provides a vivid geometric interpretation of the mutation operations that generate all Markov triples from (1,1,1). It also unifies the hyperbolic and cluster perspectives: the Markov triple graph can be seen as the Cayley graph of a certain triangle group in the hyperbolic plane.

2.1.5 Simple Geodesics, Markov Fractions, and Diophantine Approximation.

Hyperbolic geometry not only explains why Markov numbers solve an extremal approximation problem for irrational numbers, but it also sheds light on rational approximations. In 2024, Boris Springborn classified the *worst approximable rational numbers* i.e. rationals whose approximation constant is $\geq 1/3$ [12].

Springborn found that there is a planar forest of *Markov fractions* – rational numbers $\frac{p}{q}$ whose denominators q are Markov numbers – which are analogues of Markov irrationals.

In fact, these Markov fractions (between 0 and 1) can be organized in a tree (sometimes called the *Markov tree* or *Conway–Markov tree*) obtained by a modified mediant operation. Springborn proved that these fractions, together with two infinite sequences of *companions* for each (obtained by certain geodesic spiraling operations), are exactly the rationals with approximation constant $= 1/3$. Geometrically, the Markov fractions correspond to simple proper geodesic arcs on the torus (with both ends at the cusp), while their companions correspond to certain self-intersecting geodesics that avoid intersecting two particular simple geodesics.

2.1.6 Algebraic Geometry connects to hyperbolic geometry

The hyperbolic perspective also surprisingly connects to algebraic geometry. *Exceptional vector bundles* on complex surfaces provide an example: it was known from the 1980s (work of Drézet–Le Potier and Rudakov) that on the projective plane $\mathbb{P}^2$, the ranks of exceptional bundles must be Markov numbers.

Recent work by A. Veselov (2025) [8] bridges this fact with Springborn's Markov fractions. Veselov showed that, in fact, the slopes of all exceptional bundles on $\mathbb{P}^2$ are exactly the Markov fractions.

In other words, not only are the ranks constrained to Markov numbers, but the ratio c_1/r (first Chern class over rank) of each exceptional bundle is one of Springborn's special fractions. This provided a new proof of Rudakov's result (since if slope p/q is a Markov fraction, then q is a Markov number).

It also reveals an unexpected Markov property in algebraic geometry: these stable bundles are in bijection with Markov triples. This is a beautiful example of the hyperbolic/Diophantine viewpoint informing pure algebraic geometry. This aspect of the Markov numbers was discussed in Veselov's presentation *Markov fractions and the slopes of the exceptional bundles on* $\mathbb{P}^2$ at the workshop.

2.1.7 Future directions

The hyperbolic perspective leads to many intriguing questions. Markoff–Hurwitz equations are higher-dimensional analogues of the Markov equation (e.g. $x_1^2 + \cdots + x_n^2 = kx_1 \cdots x_n$ for $n > 3$). They too have an infinite tree of solutions, but the distribution of those solutions is far more complex. Zagier (1982) showed that the number of solutions up to a bound exhibits a fractional exponent (a "fractal" growth rate) in contrast to the quadratic growth for $n = 3$.

Understanding this fractal asymptotic and its geometric meaning remains an open challenge. Arthur Baragar (2025) found that curves on certain K3 surfaces have orbits with similarly puzzling growth – neither purely polynomial nor random, but fractal-like, and he conjectured a relationship between these growth-rates and the Hausdorff dimension of a certain Kleinian group that arises from consideration of the K3-surface. A tantalizing question is whether one can classify or predict the behavior of these orbits of rational points on K3 surfaces, paralleling the Markov case. These ideas were explained in Baragar's presentation *Orbits of rational points on K3 surfaces*.

Unicity has a hyperbolic interpretation: it predicts a unique simple geodesic (up to symmetries) for each length. The connection between Markov geodesics and mapping class group dynamics or modular forms is an active area of exploration (as explained by Simon's introductory talk, which proposed new results and conjectures from blending these viewpoints). The hyperbolic geometry perspective, rooted in Markov's century-old theorem, continues to drive modern research with new methods and fresh conjectures.

2.2 Markov Numbers in Number Theory

2.2.1 Classical Diophantine Results and the Unicity Conjecture.

In number theory, Markov's original interest was classifying the indefinite binary quadratic forms with minima above $1/3$. Equivalently, he described all Markov triples of positive integers via a recursive process now known as Vietá involutions.

Starting from $(1, 1, 1)$, one can generate every Markov triple by repeatedly replacing one entry using the rule $(a, b, c) \mapsto (a, b, \frac{a^2+b^2}{c})$ (and permutations thereof). This produces an infinite tree of solutions (the Markov tree). A remarkable, elementary property is that aside from the smallest two triples $(1, 1, 1)$ and $(1, 1, 2)$, all Markov triples consist of three distinct numbers. Of course, fundamental questions about the Markov numbers remain open, including notoriously *Unicity*, which was first posed by Frobenius in 1913.

Beyond Unicity, number theorists have uncovered other striking properties of Markov numbers. For example, along one branch of the tree, all Markov numbers are odd-indexed Fibonacci numbers, and along another, they are Pell numbers. Divisibility constraints show that Markov numbers avoid certain residues: no prime congruent to $3 \pmod 4$ can divide any Markov number. In particular, every odd Markov number is $1 \pmod 4$ (this fact is quite old and was known to Frobenius and Hurwitz; it is related to the fact that Markov numbers appear in solutions of $x^2 \equiv -1 \pmod p$, which requires -1 to be a quadratic residue). Another mysterious property is that Markov numbers often have few prime factors, and yet are very rarely prime powers [10]. These observations reflect the rich interplay between Markov numbers and classical number theory.

2.2.2 Structure of the Markov spectrum

The Markoff spectrum arises from studying the minima of indefinite binary quadratic forms. Given a quadratic form

$$q(x, y) = ax^2 + bxy + cy^2 \quad \text{with discriminant} \quad \Delta(q) = b^2 - 4ac > 0,$$

the Markoff constant is defined as

$$M(q) = \frac{\sqrt{\Delta(q)}}{\inf_{(x,y)\in\mathbb{Z}^2\setminus\{(0,0)\}} |q(x, y)|}.$$

The set of all such constants forms the Markoff spectrum $\mathcal{M}$. Below the value 3, this spectrum is discrete and coincides with the Lagrange spectrum. Markoff numbers, satisfying the Diophantine equation

$$m_p^2 + m_q^2 + m_r^2 = 3m_p m_q m_r,$$

index this part of the spectrum through the correspondence

$$m \longleftrightarrow \frac{\sqrt{9m^2-4}}{m}.$$

Beyond 3, the structure of the Markoff spectrum becomes more intricate. Marshall Hall showed in 1947 that the interval $[6, \infty)$, known as Hall's ray, is included in the spectrum [18]. His work relied on constructing sums of continued fractions with bounded partial quotients, such as elements of the Cantor-like set

$$C(n) = \{\alpha = [0; a_1, a_2, \ldots] \in [0,1] \mid 0 < a_i \leq n \; \forall i\}.$$

An important open problem involves the arithmetic sum $C(2) \oplus C(2)$, specifically whether it contains an interval. Although every number in $\mathcal{M} \cap [\sqrt{5}, 3)$ corresponds to a continued fraction with partial quotients in $\{1, 2\}$, this condition is not sufficient for inclusion. For example,

$$L([1; 2, 1, 2, \ldots]) = \frac{1+\sqrt{10}}{3} = \sqrt{10} \not< 3.$$

Thus, the question arises: does $C(2) \oplus C(2)$ contain any real interval? The connection between this set and $\mathcal{M} \cap [\sqrt{5}, \sqrt{12}]$ suggests a deeper structure possibly linked to the Hausdorff dimension of these sum sets.

A further extension of the problem is the study of multidimensional Markoff spectra. These generalize the classical case to forms in n variables, such as products of linear forms, and their minima over integer lattices. Very little is known about these higher-dimensional spectra, and it remains an open question whether techniques like Hall's can be generalized, or what the analogues of sets like $C(n)$ would be in this context.

Aspects of the Markoff spectrum were explored in Peter Sarnak's presentation *Diophantine analysis of Markoff type cubic surfaces*, and in the lightning talk of Luke Jeffreys.

2.2.3 Strong Approximation and Markoff Graphs mod p.

A major modern development is the study of the Markoff equation modulo primes. Consider the Markov surface $x^2 + y^2 + z^2 = 3xyz$ as an affine variety. The group of symmetries generated by Vietá involutions and permutations acts on the set of integer solutions. The *Strong Approximation Conjecture* for Markov numbers, formulated by Baragar (1991) [19] and later popularized by Bourgain, Gamburd, and Sarnak [10], posits that this action is transitive on the solutions modulo p, for every prime p (when considering nonzero solutions in $(\mathbb{Z}/p\mathbb{Z})^3$).

In graph-theoretic terms, the *Markoff graph mod* p – whose vertices are the solutions to $x^2 + y^2 + z^2 = 3xyz \pmod{p}$ (excluding the zero vector), with edges for Vietá moves – is conjectured to be connected for each prime p. This is a deep conjecture about expansion on an arithmetic graph, with links to the theory of expander graphs and sieve methods.

Significant progress has been made in the last decade. In 2016, Bourgain, Gamburd, and Sarnak (BGS) [10] proved an "almost all" result: for all but a density-zero set of primes, the Markoff graph mod p is connected. More precisely, they showed that any potential finite obstacles (primes for which the mod-p solutions split into multiple orbits) must lie in a finite (effectively computable) set. Their method combined the expansion properties of the Markoff graphs with the classification of certain algebraic $\overline{\mathbb{Q}}$-orbits.

Building on this, William Chen achieved a breakthrough in 2021: using novel tools from algebraic geometry (specifically, Hurwitz spaces of covers of elliptic curves) and nonabelian level structures, Chen proved that for all but finitely many primes p, the Markoff mod p graph is indeed connected (transitive action). Chen's result constitutes a significant step towards resolving the Strong Approximation Conjecture, reducing it to checking a finite list of small primes by direct computation. Recent work has pushed this story further. Eddy-Fuchs-Litman-Martin-Tripeny [15] have quantified Bourgain-Gamburd-Sarnak's work, demonstrating that the Markoff mod p graph is connected when $p > 10^{393}$, and Colby Brown has demonstrated numerically connectivity for $p < 10^6$ [20]. Aspects of this story were covered extensively in Colby Brown's introductory talk, in William Chen's presentation *Markoff triples mod* p *and* $\mathrm{SL}(2, p)$*-covers of elliptic curves*, and in Gamburd's presentation on Strong Approximation.

Chen's crucial breakthrough towards the Strong Approximation Conjecture was to show that connectivity of those graphs is equivalent to the connectedness of certain Hurwitz moduli spaces of $SL(2, p)$-covers of elliptic curves – and then prove that all those Hurwitz spaces are connected (again with finitely many possible exceptions). In particular, Chen showed the group of Markoff moves acts transitively on solutions mod p for all sufficiently large p, by relating the orbit count to a degree of a map between moduli stacks and proving a congruence condition on that degree. This was a tour-de-force combining arithmetic geometry and group theory.

The ramifications of Chen's theorem are significant. One immediate corollary is a *strong approximation property*: the Markoff equation's integral solutions are Zariski-dense and surject onto solutions mod p for all large primes. Another consequence is that any congruence constraints on Markov numbers must be very special (in fact, Chen's work implies that aside from the known $p \not\equiv 3 \pmod 4$ condition, there are no other systematic congruence restrictions holding for all Markov numbers).

The connectedness of Markoff graphs mod p is also related to the expansion property; conjecturally, these graphs form a family of expanders, which has implications in complexity theory and combinatorics. Very recently, Daniel Martin (2025) gave an alternate proof of a special case of Chen's result: he proved that in any connected component of the Markoff graph mod p, the number of vertices is divisible by p. This proof was presented by Daniel Martin in his talk *Arithmetic in Markoff mod p subgraphs*.

2.2.4 Diophantine and Algebraic Open Questions.

Of course, though there are at most finitely many exceptions, the Strong Approximation Conjecture remains open. There are also other number-theoretic challenges. The Unicity Conjecture remains the paramount open problem, and while there are recent advances towards understanding variations of Markov numbers in the Farey tree, a full proof likely requires a new idea. Another open direction is the Markoff-Hurwitz equations for $n > 3$: number theorists have shown these higher-dimensional analogues do not enjoy the same kind of uniqueness or monotonicity, and the count of solutions grows in a fractal way as mentioned above. Understanding the distribution of fractal Markov numbers for $n \geq 4$ is largely uncharted. Also of interest is the appearance of Markov numbers in other Diophantine equations and combinatorial contexts – for example, Markov numbers appear in relation to Frobenius coin-exchange problems and in the theory of Apollonian circle packings (there is an analogy between the Markov tree and Descartes' circle equation).

2.3 Markov Numbers in Cluster Algebras

2.3.1 Cluster Algebra of the Once-Punctured Torus

Markov numbers also appear in the realm of *cluster algebras* – a framework introduced by Sergey Fomin and Andrei Zelevinsky in 2002 to study recursive combinatorial structures in algebra [13].

The connection arises from a simple observation: the Markov triple recurrence $(x, y, z) \mapsto (x, y, 3xy-z)$ (the Vietá involution) is analogous to a cluster mutation. In fact, if one considers a *rank-3 cluster algebra* with initial exchange matrix (quiver) forming a triangle (often called the *Markov quiver* with three vertices), the exchange relation can be written in the form $z' = \frac{x^2+y^2}{z}$, which mirrors the Markov equation. Specifically, the cluster algebra associated to a once-punctured torus has an exchange relation that is exactly the Markov equation when specialized appropriately.

James Propp noted in 2005 that Markov numbers appear as a specialization of the cluster variables in this cluster algebra [5]. Independently, Beineke, Brüstle, and Hille [22] showed that the Markov equation describes the exchange relations for a cluster-cyclic quiver with three vertices. In summary, Markov triples are in bijection with seeds (clusters) in the once-punctured torus cluster algebra, and the Markov mutations correspond to cluster mutations.

One concrete way to see this is: label the three cluster variables in the initial seed as x, a, b on a triangle quiver. The exchange relation for the variable x (mutating at x) will produce a new variable x' satisfying $x \cdot x' = a^2 + b^2$ (assuming a certain convention of exchange matrix). If we set the initial cluster variables $(x, a, b) = (1, 1, 1)$ (all ones), then by induction each subsequent cluster variable in this algebra takes an integer value – in fact, those values are exactly the Markov numbers. All Markov numbers can be generated by evaluating the cluster variables of the once-punctured torus algebra at the initial seed values 1. This situates the Markov triple recursion in a broader algebraic context, allowing one to apply general cluster algebra techniques (such as g-vectors, F-polynomials, etc.) to study Markov numbers. This viewpoint was discussed in Dani Kaufman's talk *Non-commutative Markov numbers* and in the lightning talk of Sam Evans *Arithmetic and geometry of Markov polynomials*.

2.3.2 Snake Graphs and Perfect Matchings.

A powerful combinatorial tool in cluster algebras is the *snake graph* technique (developed by Musiker, Schiffler, et al.). For cluster algebras from surfaces, each cluster variable can be associated with a snake graph, a certain planar graph whose perfect matchings correspond to terms in that cluster variable's Laurent expansion. In the case of the Markov (punctured torus) algebra, every cluster variable can be obtained as the number of perfect matchings of a certain snake graph. When the cluster variables are specialized at 1, this number of perfect matchings is exactly the cluster variable's value.

Therefore, each Markov number is the number of perfect matchings of a corresponding snake graph. Ralf Schiffler and collaborators exploited this combinatorial interpretation to great effect. In a joint work, Lee–Li–Rabideau–Schiffler [9] studied monotonicity properties of Markov numbers by examining how these snake graphs grow when "tilted" along a fixed slope. They proved that along any line of slope between $-8/7$ and $-5/4$, the Markov numbers increase or decrease monotonically. In particular, the conjectured orderings for slope 0 and 1 (Aigner's conjectures) follow as special cases. The snake graph model was crucial in their proof, reducing inequalities between Markov numbers to inclusion relations between perfect matching sets. By combinatorial means, they circumvented difficulties that had stymied direct number-theoretic approaches for years. The resolution of the three Aigner conjectures thus stands as a triumph for the cluster algebra perspective on Markov numbers.

2.3.3 Cluster Geometry and Non-Commutative Markov Numbers.

The relationship between the cluster algebra and hyperbolic geometry is also a two-way street. On one hand, Felikson's work (mentioned above) showed how geometric group theory can interpret cluster mutations. On the other hand, cluster algebras themselves have geometric realizations (via Teichmüller spaces and lambda lengths). The once-punctured torus cluster algebra corresponds to the decorated Teichmüller space of a one-cusped torus, which is essentially the hyperbolic modular torus – hence the same object underlying Markov's classical theory. This provides a direct bridge between the cluster Y-variables (which satisfy a periodicity known as the Markov moves) and the length spectra of geodesics. Thereby, cluster algebra techniques might offer a path to proving uniqueness or other metric properties by translating them into algebraic identities.

Another exciting development is the extension of Markov's story to non-commutative algebras. Recently, Dani Kaufman, Zachary Greenberg, and Anna Wienhard applied the framework of non-commutative cluster algebras (due to Berenstein and Retakh) to the Markov equation [16]. In their work, they define a *non-commutative Markov equation* in a ring with involution, by evaluating the cluster variables of the non-commutative cluster algebra of a once-punctured torus at appropriate (matrix or ring) values.

This yields non-commutative Markov numbers, which are elements of the chosen ring that reduce to the usual Markov numbers upon taking a commutative specialization. For example, by plugging in certain 2×2 matrices or dual numbers for the initial cluster data, they constructed "Markov numbers" that are polynomials (deformations of the classical integers), dual number analogues, or elements of a group algebra.

These provide new invariants and possibly symmetries of the Markov equation. While still in early stages, this non-commutative generalization opens the door to considering Markov-type equations in settings like free groups or quantum clusters, where one might define a quantum Markov equation. It also enriches the algebraic structure: for instance, one can ask if the non-commutative Markov numbers in a group ring encode the classical Markov primes or any subtle congruence information. Kaufman's talk indicated a variety of examples and a general framework to produce such exotic Markov "numbers" in arbitrary rings. This is a fresh avenue of research at the intersection of cluster algebras, algebraic geometry, and even physics (since cluster algebras have appeared in quantum dilogarithm identities and Teichmüller theory).

2.3.4 Future directions

The cluster algebra point of view has proven extremely fruitful for combinatorial and algebraic aspects of Markov numbers. It transformed understanding of the Markov tree into understanding of a cluster complex, where tools like g-vectors and seed mutations translate into Diophantine properties of Markov triples [9].

With all three of Aigner's monotonicity conjectures now resolved by cluster methods, one might be tempted to attack the full unicity conjecture using cluster algebra as well. However, the unicity conjecture is equivalent to saying that no two distinct sequences of cluster mutations (starting from the initial seed) ever produce the same cluster variable value as the largest element. This seems to require a very strong form of separation between mutation paths. While not yet achieved, cluster theory provides a clear language to formulate this: it would mean the cluster fan for the Markov quiver has a certain property ensuring a unique maximal g-vector for each cluster variable value. This remains difficult. Nonetheless, the progress so far suggests that the blend of combinatorics (snake graphs), algebra (Laurent polynomials), and even geometry (tropical and Teichmüller interpretations) inherent in cluster algebras will continue to shed light on Markov numbers. The fact that a 19th-century Diophantine problem finds natural expression in a 21st-century algebraic theory is a testament to the unity of mathematics – and it hints that further surprises may be in store as these perspectives converge.

3 Presentation Highlights

Here is a list of presentations delivered at the workshop:

- *Intro 1 - Markov numbers and hyperbolic geometry* by Christopher-Lloyd Simon, Monday 9:00-10:00
- *Intro 2 - Markov numbers and number theory* by Colby Brown, Monday 10:30-11:30
- *Intro 3 - Markov numbers and cluster algebras* by Ryan Schroeder, Monday 13:00-14:00
- *Diophantine analysis of Markoff type surfaces* by Peter Sarnak, Monday 15:00-16:00
- *The worst approximable rational (sic!) numbers* by Boris Springborn, Tuesday 9:00-9:30
- *Markov fractions and the slopes of the exceptional bundles on* $\mathbb{P}^2$ by Alexander Veselov, Tuesday 9:45-10:15
- *Strong Approximation* by Alexander Gamburd, Tuesday 10:45-11:15
- *Markoff triples mod p and SL(2,p)-covers of elliptic curves* by William Chen, Tuesday 11:30-12:00
- *Arithmetic in Markoff mod p subgraphs* by Daniel Martin, Tuesday 13:45-14:15
- *Monotonicity of Markov numbers via perfect matchings of snake graphs* by Ralf Schiffler, Tuesday 14:30-15:00
- *Groups generated by three symmetries on the hyperbolic plane* by Anna Felikson, Tuesday 15:45-16:15
- *Fricke's trace identity and spin groups* by Matthew de Courcy-Ireland, Tuesday 16:30-17:00
- Lightning session talks, Tuesday 20:00-21:00
  - *Arithmetic and geometry of Markov polynomials* by Sam Evans
  - *Markov's conjecture on integral necklaces* by David Fisac
  - *Residual Transitivity modulo p implies Minimality (for Markoff surfaces)* by Seung uk Jang
  - *Lagrange and Markoff spectra* by Luke Jeffreys
- *Orbits of rational points on $K3$ surfaces* by Arthur Baragar, Thursday 9:00-9:30
- *Non-commutative Markov numbers* by Dani Kaufman, Thursday 10:00-11:00
- *A new proof of the Markov theorem* by Ian Agol, Thursday 11:15-11:45
- *Twist tori equidistribute in moduli space* by Aaron Calderon, Thursday 13:45-14:15
- *Orbifold Markov numbers* by Esther Banaian, Thursday 14:30-15:00

- *Markov numbers, Fock's function, and Mather's β function* by Alfonso Sorrentino, Thursday 15:45-16:15
- *Markoff triples and linear recurrence sequences* by Elisa Bellah, Thursday 16:30-17:00

Wednesday and Friday mornings consisted of participants breaking into groups to pose research questions, and consider new directions for exploration. A list of topics discussed can be found in the following section.

4 Scientific Progress Made

We briefly describe some of the questions and open problems proposed and explored during the workshop. Some of these questions have become active current collaborations.

4.1 Hyperbolic geometry

1. (Simon/Gaster/Martinez-Granado) Understand closed simple geodesics and simple proper geodesic arcs in quotients of congruence subgroups $\Gamma(2), \Gamma(3), \Gamma(5)$. In particular, what are their length / λ-length spectra? What are the arithmetic properties of these numbers? In a similar vein, identify the fractions p/q that project to simple arcs on the hyperbolic surface which is the commutator cover of the modular torus. Conjecturally, this set of denominators is given by $\mathbb{N}$ – this would in particular imply the Zaremba Conjecture; is it possible to count the number of simple arcs with λ-length at most n? prove density of the simple λ-lengths in $[0, n]$? Gaster, Martinez-Granado and Gaster plan to continue thinking on these questions together.
2. (Springborn) The geometric interpretation of Markov numbers comes from looking at the hexagonal torus. Is there an analogous theory for the square torus?
3. (Agol) The Markov spectrum above 3 has many gaps, until at about 4.5, Freiman's constant. Find topological explanation for geodesics below Freiman's constant: do closed geodesics which don't go deep into the cusp have few bigons?
4. (Gaster) The Unicity Conjecture is hard, but the geometric perspective might at least yield new bounds on the multiplicity of the Markov number n (or, equivalently, simple closed geodesics of length equal to $L \approx \log(n)$). For example, using a bound by Jarnik for integral points on strictly convex curves on the plane, one can get an upper bound of $\log(n)^{2/3}$. Can we use our finer understanding of the space of meausured laminations to get better bounds?
5. (Gaster/Martinez-Granado) Same question as above, but using Teichmüller dynamics, in particular, earthquake flow excursions.
6. (Gaster/Martinez-Granado) From the number theoretic viewpoint, Markov numbers can be associated to minima of integral primitive indefinite binary quadratic forms. In fact, there is a correspondence between geodesics in the modular surface and such quadratic forms up to $PSL_2(\mathbb{Z})$ conjugacy. Given two quadratic forms of the same discriminant (same length closed geodesics), there is a well-defined composition operation (up to conjugacy), $q_1 * g_2 = q_3$. Is there a geometric characterization of when q_1, q_2, q_3 appear in such a triple?
7. (Chen) Can one get better bounds on the size of components of $\mod p$ graph by using hyperbolic geometry and length estimates in the Farey tree?
8. (Fuchs) How many "k-companion" Markov numbers (a notion developed by Springborn, and explained in his talk) are there $\leq N$?
9. (Gaster) What bounds are possible for the number of simple closed geodesics of length precisely L on a closed hyperbolic surface of given topological type? Schmutz-Schaller conjectures that, for punctured tori, the Markov bound of 6 holds independent of hyperbolic structure. Try to find constructions of many such curves.

10. (de Courcy-Ireland/Jeffreys) A family of graphs studied by Jeffreys, related to square-tiled surfaces has features paralleling de Courcy-Ireland's work on Markoff graphs modulo prime numbers. Study their Euler characteristic and the bulk distribution of their eigenvalues.

4.2 Cluster algebras

1. Prove or disprove that all solutions in integer Laurent polynomials of $X^2+Y^2+Z^2 = k(x,y,z)XYZ$, where $k(x,y,z) = \frac{x^2+y^2+z^2}{xyz}$, can be found from $X = x$, $Y = y$, $Z = z$ by mutations, generalizing the situation for the Markov equation.

2. (Banaian/Yildirim) Investigate the connection of Markov numbers with the representation theory of quivers and cluster algebras. First of all, a possible representation theory connection we want to delve into is inspired by the paper by Alex Lasnier titled "Christoffel words and Markoff triples: an algebraic approach" (https://arxiv.org/pdf/1104.1799). The author in that paper uses certain string modules, called Markoff modules, to get Markov numbers/triples by defining a binary tree isomorphism. Moreover, they define a mutation to get from one Markov triple to another by using using approximating triangles. We would like to understand this construction in a more categorical way: By understanding these Markoff modules as objects in certain cluster categories and their mutation as the mutation in the cluster categories. We suspect that this mutation Lasnier defines should coincide with the mutation of quiver with potential in the sense of Daniel Labardini-Fragoso (https://arxiv.org/abs/1302.1936).

   Moreover, there is a one-to-one correspondence between Markov triples and Christoffel words. There are also celebrated bijections between elements of Coxeter groups and certain classes in the representation theory of preprojective algebras. See for instance the paper by by Yuya Mizuno and Hugh Thomas (https://arxiv.org/pdf/1804.02148). One interesting question is to combine all these beautiful mathematics and getter a better understanding of Markov triples in this type of representation theory. Since Coxeter groups are quite broad, this may lead to new discoveries for Markov numbers or some new connections to generalized Markov triples.

3. (de Courcy-Ireland/Kaufman) In Kaufman's paper with Greenberg and Wienhard, Kaufman proves trace relations for SL_2 over non-commutative rings. Taking SL_2 over a ring of matrices gives an identity for the symplectic group. de Courcy-Ireland had also encountered this symplectic identity in a project on spin groups. There might be other connections, and de Courcy-Ireland plans to explore them further. Furthermore, they anticipate that Martin's approach to Chen's congruence might give congruences for the exchange graphs of other cluster algebras.

4.3 Number theory

1. (Chopra) What is the probability that the first digit of a Markoff number is 1? Is it $\log_{10}(2)$? (and is the probability that the first digit is d equal to $\log_{10}(1+\frac{1}{d})$?

2. (Brown/Littman) Is $(1,1,1)$ in the 'big component' of the Markoff $\mod p$ graph (explained in the work of Bourgain-Gamburd-Sarnak, and elucidated by Chen, Fuchs, Littman, de Courcy-Ireland, etc.)?

3. (Baragar) It is well-known that Unicity holds for Markov numbers m such that $m = p^k$ for prime p, and yet Bourgain-Gamburd-Sarnak showed that most Markoff numbers are composite. It is also well-known that Unicity holds for Markov numbers m such that $3m \pm 2 = p^k$ for prime p. Is it true that $3m \pm 2$ is composite for most Markoff numbers?

4. (?) Let M_n indicate the nth Markoff number. Provide upper or lower bounds for $M_{n+1} - M_n$. Can $\sup_{n \leq N} M_{n+1} - M_n$ be bounded by a polynomial in N? (or even sub-polynomial?)

5. (Simon) Two irrational numbers $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ are *eventually* $\mathrm{PGL}_2(\mathbb{Z})$-*equivalent* (that is, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, the numbers $n\alpha$ and $n\beta$ belong to the same $\mathrm{PGL}_2(\mathbb{Z})$-orbit) if and only if $\alpha \equiv \pm\beta \pmod 1$. At the time of finishing this report, there has been partial progress towards this conjecture, in recent work by Schmiedling and Simon [2, Proposition A.8]. Daniel Martin and Simon plan to continue exploring this conjecture together.

6. (Gaster/Martinez-Granado) The class numbers groups associated to determinants containing quadratic forms associated to Marov numbers seem to display some interesting structure. Understand this patterns, specifically, can one characterize the role of the Markov element in each class group. Gaster, Martinez-Granado and Simon plan to continue this project together.
7. (Litman/de Courcy-Ireland) Study the orbits mod p for the Markoff-type surfaces studied by Gyoda and Matsushita. Integer points on those same surfaces appeared in Esther Banaian's talk. Investigate further what can be said about strong approximation or local-global obstructions for these surfaces.

5 Outcome of the Meeting

The workshop successfully fostered vibrant interdisciplinary dialogue among researchers in hyperbolic geometry, number theory, and cluster algebras. This cross-pollination of ideas led to new collaborations, as evidenced by joint projects initiated during the meeting—such as those between de Courcy-Ireland and Jeffreys, and between Martínez-Granado and Simon, among many others.

The meeting also served as an entry point into current research for graduate students, who were introduced to a wide array of open problems from multiple perspectives. In addition, it provided a valuable platform for early-career researchers to present their work—whether through short presentations, in-depth introductory talks, or lightning talks. For example, Fisac and Jang gave lightning talks as graduate contributors. Brown gave an introductory lecture on Markov numbers from the number-theoretic standpoint, and Evans delivered a short talk on his research.

Overall, the meeting created a stimulating environment that deepened cross-disciplinary connections and broadened participation among junior researchers.

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