Symmetric Monge-Kantorovich problems and polar decompositions of vector fields

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Abstract

For any given integer $N \geq 2$, we show that every bounded measurable vector field from a bounded domain Ω into $\mathbb{R}^d$ is N-cyclically monotone up to a measure preserving N-involution. The proof involves the solution of a multidimensional symmetric Monge-Kantorovich problem, where the cost function on the product domain Ω^N is given by the vector field (actually $N-1$ of them). We show that the supremum over all probability measures on Ω^N which are invariant under cyclic permutations and with a given first marginal μ, is attained on a probability measure that is supported on the graph of a function of the form $x \to (x, Sx, S^2x, ..., S^{N-1}x)$, where S is a μ-measure preserving transformation on Ω such that $S^N = I$ a.e. The proof exploits a remarkable duality between such involutions and those Hamiltonians that are N-cyclically antisymmetric.

1 Introduction

Given a probability measure μ on a domain Ω in $\mathbb{R}^d$, that is absolutely continuous with respect to Lebesgue measure, and a bounded above upper semi-continuous cost function $c(x_1, x_2, ..., x_N)$ on Ω^N, we consider the following symmetric Monge-Kantorovich problem

$$\mathrm{MK}^1_{\mathrm{sym}}(c) = \sup\left\{\int_{\Omega^N} c(x_1, x_2, ..., x_N)d\pi;\ \pi \in \mathcal{P}_{sym}(\Omega^N, \mu)\right\} \tag{1}$$

where $\mathcal{P}_{sym}(\Omega^N, \mu)$ denotes the set of all probability measures on Ω^N, whose marginals are equal to μ and which are invariant under the cyclical permutation

$$\sigma(x_1, x_2, ..., x_N) = (x_2, x_3, ..., x_N, x_1).$$

In other words, $\pi \in \mathcal{P}_{sym}(\Omega^N, \mu)$ if

$$\textstyle\int_{\Omega^N} f(x_1, x_2, ..., x_N)d\pi = \int_{\Omega^N} f(\sigma(x_1, x_2, ..., x_N))d\pi \text{ for every } f \in C(\Omega^N), \tag{2}$$

and

$$\textstyle\int_{\Omega^N} f(x_i)d\pi = \int_{\Omega} f(x_i)d\mu \text{ for every } f \in C(\Omega). \tag{3}$$

Standard results show that there exists $\pi_0 \in \mathcal{P}_{sym}(\Omega^N, \mu)$ where the supremum above is attained. In this paper, we are interested in an important class of cost functions c, where the optimal measure π_0 is necessarily supported on the graph of a function of the form $x \to (x, Sx, S^2x, ..., S^{N-1}x)$, where S is a μ-measure preserving transformation on Ω such that $S^N = I$ a.e.

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If c is finite, then one can extend the original approach of Kantorovich to the multi-marginal and cyclically symmetric case to show that (1) is dual to the following minimization problem

$$\mathrm{DK}^1_{\mathrm{sym}}(c) := \inf\Big\{N\int_\Omega u(x)\,d\mu;\; u:\Omega\to\mathbb{R} \text{ and } \sum_{j=1}^{N} u(x_j) \geq \frac{1}{N}\sum_{i=0}^{N-1} c(\sigma^i(x_1,\dots,x_N))\Big\}. \tag{4}$$

In this paper, we introduce a new dual problem based on the class $\mathcal{H}_N(\Omega)$ of all N-*cyclically antisymmetric Hamiltonians* on Ω^N, that is

$$\mathcal{H}_N(\Omega) = \{H \in C(\Omega^N;\mathbb{R});\; \textstyle\sum_{i=0}^{N-1} H\left(\sigma^i\left(\mathbf{x}\right)\right) = 0 \text{ for all } \mathbf{x}\in\Omega^N\}. \tag{5}$$

We shall say that H is N-*sub-antisymmetric on* Ω if

$$\sum_{i=0}^{N-1} H(\sigma^i(x_1,...,x_N)) \leq 0 \text{ on } \Omega^N. \tag{6}$$

For $H\in\mathcal{H}_N(\Omega)$, we let ℓ^c_H be the "c-Legendre transform" of H with respect to the last $(N-1)$ variables, i.e.,

$$\ell^c_H(x) = \sup\left\{c(x,x_2,...,x_N) - H\left(x,x_2,...,x_N\right);\left(x_2,...,x_N\right)\in\Omega^{N-1}\right\},$$

and consider the problem

$$\mathrm{DK}^2_{\mathrm{sym}}(c) := \inf\left\{\int_\Omega \ell^c_H(x)d\mu(x);\; H\in\mathcal{H}_N(\Omega)\right\}. \tag{7}$$

Since $\int_\Omega H(x_1,x_2,...,x_N)\,d\pi \leq 0$ for each $H\in\mathcal{H}_N(\Omega)$ and any symmetric probability π on Ω^N, we have for each $\pi\in\mathcal{P}_{sym}(\Omega^N,\mu)$

$$\int_{\Omega^N} c(x_1,x_2,...,x_N)d\pi \leq \int_{\Omega^N}\left[c(x_1,x_2,...,x_N) - H(x_1,x_2,...,x_N)\right]d\pi \leq \int_\Omega \ell^c_H(x_1)\,d\mu(x_1),$$

and therefore $\mathrm{MK}^1_{\mathrm{sym}}(c)\leq \mathrm{DK}^2_{\mathrm{sym}}(c)$.

Of great interest is to determine for which cost functions c, there is no duality gap, that is

$$\mathrm{MK}^1_{\mathrm{sym}}(c) = \mathrm{DK}^1_{\mathrm{sym}}(c) = \mathrm{DK}^2_{\mathrm{sym}}(c). \tag{8}$$

In this paper, we shall focus on cost functions on Ω^N of the form

$$c(x_1,x_2,...,x_N) = \langle u_1(x_1),x_2\rangle + ...\langle u_{N-1}(x_1),x_N\rangle, \tag{9}$$

where $u_1,...,u_{N-1}$ are given vector fields from Ω to $\mathbb{R}^d$. In this case,

$$\ell_H(x) = \sup\left\{\langle u_1(x),x_2\rangle + ...\langle u_{N-1}(x),x_N\rangle - H(x,x_2,...,x_N);\;(x_2,...,x_N)\in\Omega^{N-1}\right\},$$

which means that ℓ_H is essentially the standard Lagrangian associated to H (i.e., Legendre transform of H with respect to the last $N-1$-variables) and

$$\ell_H(x) = L_H(x,u_1(x),u_2(x),...,u_{N-1}(x)),$$

where for $(x,p_1,...,p_{N-1})\in(\mathbb{R}^d)^N$,

$$L_H(x,p_1,...,p_{N-1}) = \sup\{\sum_{i=1}^{N-1}\langle p_i,y_i\rangle - H(x,y_1,...,y_{N-1});y_i\in\Omega\}.$$

We are also interested in the "extremal" probability measure in $\mathcal{P}_{sym}(\Omega^N,\,\mu)$, which are images of μ by maps of the form $x\to(x,Sx,S^2x,...,S^{N-1}x)$, where S is a μ-preserving transformation such that $S^N=I$, μ a.e.

For that, we consider the set $\mathcal{S}(\Omega)$ of measure preserving transformations on Ω, which can be considered as a closed subset of the sphere of $L^2(\Omega, \mathbb{R}^d)$ and set

$$\mathcal{S}_N(\Omega) = \{S \in \mathcal{S}(\Omega), S^N = I \text{ a.e.}\}$$

The set $\mathcal{S}_N(\Omega)$ has been shown recently in [3] to be polar to the class of *N-cyclically monotone vector fields*, which are those $u : \Omega \to \mathbb{R}^d$ that satisfy for every cycle $x_1, ..., x_N, x_{N+1} = x_1$ of points in Ω, the inequality

$$\sum_{i=1}^{N} \langle u(x_i), x_i - x_{i+1} \rangle \geq 0. \tag{10}$$

The following theorem, which is the main result of this paper points –among other things– to the close connection between these four fundamental notions of modern analysis.

Theorem 1.1 *Given $(N-1)$ bounded vector fields $u_1, u_2, ..., u_{N-1}$ from Ω to $\mathbb{R}^N$, and a probability measure μ on Ω that is absolutely continuous with respect to Lebesgue measure, we consider the following variational problems:*

$$\text{MK} : \;=\; \sup\{\int_{\Omega^N} [\langle u_1(x_1), x_2\rangle + ...\langle u_{N-1}(x_1), x_N\rangle]\, d\pi; \pi \in \mathcal{P}_{sym}(\Omega^N, \mu)\} \tag{11}$$

$$\text{DK} : \;=\; \inf\{\int_{\Omega} L_H(x, u_1(x), u_2(x), ..., u_N(x))\, d\mu(x); H \in \mathcal{H}_N(\Omega)\}. \tag{12}$$

$$\text{MK}' : \;=\; \sup\{\int_{\Omega^N} \left[\langle u_1(x), Sx\rangle + \langle u_2(x), S^2x\rangle + ...\langle u_{N-1}(x), S^{N-1}x\rangle\right] d\mu;\ S \in \mathcal{S}_N(\Omega)\}. \tag{13}$$

If $\text{meas}(\partial\Omega) = 0$*, then the following holds:*

1. *MK=DK=MK'.*
2. *MK' is attained at some $S \in \mathcal{S}_N(\Omega)$, which means that MK is attained at an invariant measure π_S that is the image of μ by the map $x \to (x, Sx, S^2x, ..., S^{N-1}x)$.*
3. *There exists a function H on $\mathbb{R}^{dN}$ that is concave in the first variable, convex in the last $(N-1)$ variables and N-sub-antisymmetric on Ω, such that*

$$(u_1(x), ..., u_{N-1}(x)) \in \partial_{2,...,N} H(x, Sx, ..., S^{N-1}x) \quad a.e.\ x \in \Omega. \tag{14}$$

   Moreover, if either $u_i \in W^{1,1}_{loc}(\Omega)$ for $i = 1, 2, ..., N-1$ or if S is differentiable a.e., then there exists a N-cyclically antisymmetric Hamiltonian $H \in \mathcal{H}_N(\Omega)$ such that

$$(u_1(x), ..., u_{N-1}(x)) = \nabla_{2,...,N} H(x, Sx, ..., S^{N-1}x) \quad a.e.\ x \in \Omega. \tag{15}$$

4. *Assume that for any two families of points $x_1, ..., x_N$ and $y_1, ..., y_N$ in Ω, the function*

$$x \to \sum_{i=1}^{N-1} \langle u_i(x), y_i - x_i\rangle + \sum_{i=1}^{N-1} \langle u_i(y_{N-i}) - u_i(x_{N-i}), x\rangle$$

   has no critical point unless when $x_1 = y_1$. Then there exists a unique measure preserving N-involution S such that (15) *holds for some concave-convex N-sub-antisymmetric Hamiltonian H.*

If $u : \Omega \to \mathbb{R}^d$ is a single bounded vector field, then the above theorem applied to the family $(0, ..., 0, u)$ yields the decomposition

$$(-u(Sx), 0, ..., 0, u(x)) = \nabla H(x, Sx, ..., S^{N-1}x) \qquad a.e. \quad x \in \Omega. \tag{16}$$

If S is the identity in the above representation, it is then easy to see that u is N-cyclically monotone, which means that the above theorem essentially says that any bounded vector field is N-cyclically monotone up to a measure preserving N- involution. This is clearly in the same spirit as Brenier's theorem stating that any non-degenerate vector field is the gradient of a convex function (i.e., is N-cyclically monotone for all N) modulo a measure preserving transformation. Note that the representation of 2-monotone operators as partial gradients of antisymmetric saddle functions was established by Krause [9]. The general version of this result was established in [8] where it is shown that any bounded vector field is 2-monotone up to a measure preserving involution. Theorem 1.1 can be see as an extension of this result to the case where $N \geq 2$ and where there is more than one vector field.

Actually, in the case of a single vector field $u : \Omega \to \mathbb{R}^d$, one need not consider Hamiltonians on Ω^N as long as the requirement of N-antisymmetry is replaced by the following property: Say that a function F on $\mathbb{R}^d \times \mathbb{R}^d$ is *N-cyclically sub-antisymmetric on* Ω, if

$$F(x,x) = 0 \text{ and } \sum_{i=1}^{N} F(x_i, x_{i+1}) \leq 0 \text{ for all cyclic families } x_1, ..., x_N, x_{N+1} = x_1 \text{ in } \Omega. \tag{17}$$

Note that if a function $H(x_1, ..., x_N)$ is N-sub-antisymmetric and if it only depends on the first two variables, then the function $F(x_1, x_2) := H(x_1, x_2, ..., x_N)$ is N-cyclically sub-antisymmetric.

Our proof then yields the following result.

Theorem 1.2 *Consider a vector field $u \in L^\infty(\Omega, \mathbb{R}^d)$, then:*

1. *For every $N \geq 2$, there exists a measure preserving N-involution S on Ω and a globally Lipschitz concave-convex function F of $\mathbb{R}^d \times \mathbb{R}^d$ that is N-cyclically sub-antisymmetric on Ω, such that*

$$(-u(Sx), u(x) \in \partial F(x, Sx) \text{ for a.e. } x \in \Omega, \tag{18}$$

where ∂H is the sub-differential of H as a concave-convex function [13].

2. *If either $u \in W^{1,1}_{loc}(\Omega)$ or if S is differentiable a.e., then*

$$u(x) = \nabla_2 F(x, Sx) \text{ for a.e. } x \in \Omega. \tag{19}$$

3. *Moreover u is strictly N-cyclically monotone on Ω if and only if $S = I$ in the representation (19).*

Remark 1.3 Note that we cannot expect to have a function F such that $\sum_{i=1}^{N} F(x_i, x_{i+1}) = 0$ for all cyclic families $x_1, ..., x_N, x_{N+1} = x_1$ in Ω. This is the reason why one needs to consider functions of N-variables in order to get N-antisymmetry as opposed to sub-antisymmetry. Note that the function defined by

$$H(x_1, x_2, ..., x_N) := \frac{(N-1)F(x_1, x_2) - \sum_{i=2}^{N-1} F(x_i, x_{i+1})}{N}, \tag{20}$$

is N-antisymmetric in the sense of belonging to $\mathcal{H}_N(\Omega)$ and $H(x_1, x_2..., x_N) \geq F(x_1, x_2)$ on Ω^N.

2 Duality between monotonicity, cyclical symmetry and involutions

We first state here a recent result of Galichon-Ghoussoub [3], which establishes the remarkable duality between N-cyclically monotone operators, N-antisymmetric Hamiltonians and measure preserving N-involutions. These dualities originated in the work of Krause on monotone operators (i.e., when $N = 2$) and the celebrated result of Brenier on the Monge transportation problem. The following notion considered recently by Galichon-Ghoussoub [3] turned out to be the appropriate extension to when $N \geq 3$.

Definition 2.1 *A family of vector fields $u_1, u_2, ..., u_{N-1}$ from $\Omega \to \mathbb{R}^d$ is said to be jointly N-monotone if for every cycle $x_1, ..., x_{2N-1}$ of points in Ω such that $x_{N+l} = x_l$ for $1 \leq l \leq N-1$, we have*

$$\sum_{i=1}^{N} \sum_{l=1}^{N-1} \langle u_l(x_i), x_i - x_{i+l} \rangle \geq 0. \tag{21}$$

Note that if each u_ℓ is N-cyclically monotone, then the family $(u_1, u_2, ..., u_{N-1})$ is jointly N-monotone. Actually, one needs much less, since the $(N-1)$-tuplet $(u, u, ..., u)$ is jointly N-monotone if and only if u is 2-monotone. On the other hand, $(u, 0, 0, ..., 0)$ is jointly N-monotone if and only if u is N-monotone. See [3] for a complete discussion.

Theorem 2.2 (Galichon-Ghoussoub) *Let $u_1, ..., u_{N-1} : \Omega \to \mathbb{R}^d$ be bounded measurable vector fields. The following properties are then equivalent:*

1. *The family $(u_1, ..., u_{N-1})$ is jointly N-monotone a.e., that is there exists a measure zero set Ω_0 such that $(u_1, ..., u_{N-1})$ is jointly N-monotone on $\Omega \setminus \Omega_0$.*
2. *The family $(u_1, ..., u_{N-1})$ is in the polar of $\mathcal{S}_N(\Omega, \mu)$ in the following sense,*
$$\inf \left\{ \int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), x - S^\ell x \rangle \, d\mu; S \in \mathcal{S}_N(\Omega, \mu) \right\} = 0. \tag{22}$$
3. *There exists a N-sub-antisymmetric Hamiltonian H which is concave in the first variable, convex in the last $(N-1)$ variables such that*
$$(u_1(x), ..., u_{N-1}(x)) = \nabla_{2,...,N} H(x, x, ..., x) \quad \text{for a.e. } x \in \Omega. \tag{23}$$
Moreover, H is N-cyclically antisymmetric in the following sense: For a.e. $\mathbf{x} = (x_1, ..., x_N) \in \Omega^N$, we have
$$H(x_1, x_2, ..., x_N) + H_{2,...,N}(x_1, x_2, ..., x_N) = 0$$
where $H_{2,...,N}$ is the concavification of the function $K(\mathbf{x}) = \sum\limits_{i=1}^{N-1} H(\sigma^i(\mathbf{x}))$ with respect to the last $N-1$ variables.

Note that (22) shows that the above is also equivalent to the statement that

$$\sup\{\int_{\Omega^N} \sum_{\ell=1}^{N-1} \langle u_\ell(x_1), x_{\ell+1} \rangle d\pi(\mathbf{x});\ \pi \in \mathcal{P}_{\rm sym}(\Omega^N, \mu)\} = \int_\Omega \sum_{\ell=1}^{N-1} \langle u_\ell(x), x \rangle \, d\mu(x), \tag{24}$$

and that the supremum is attained at the image of μ by the map $x \to (x, x, ..., x)$, which is nothing but a particular case of the symmetric Monge-Kantorovich problem, when the cost function is the one we are considering in (9) and when the family $(u_1, ..., u_{N-1})$ is N-monotone. Theorem 1.1 now appears as the extension of the above to an arbitrary family of $(N-1)$ vector fields.

At the heart of our results, is the fact that the duality between N-antisymmetric Hamiltonians and measure preserving N-involutions can be significantly strengthened from the one noted in [3]. The following lemma will be crucial to what follows.

Lemma 2.3 *Let $S_1, S_2, ..., S_{N-1}$ be μ-measurable maps on Ω. The following statements are then equivalent:*

1. *$\int_\Omega H(x, S_1 x, S_2 x, ..., S_{N-1} x) d\mu(x) = 0$ for every N-cyclically antisymmetric Hamiltonian H.*
2. *There exists a μ-measure preserving transformation $S : \Omega \to \Omega$, such that $S^N = I$ and $S_i = S^i$ for all $i = 1, ..., N-1$.*

Proof: If S is μ-measure preserving and $S^N = I$ a.e., then

$$\int_\Omega H(x, Sx, S^2x, ...S^{N-1}x) d\mu = \int_\Omega H(Sx, S^2x, ...S^{N-1}x, x) d\mu(x) = ... = \int_\Omega H(S^{N-1}x, ..., S^2x, x, Sx) d\mu(x)$$

Since H is N-antisymmetric, then

$$H(x, Sx, S^2x, ...S^{N-1}x) + H(Sx, S^2x, ...S^{N-1}x, x) + ...H(S^{N-1}x, ..., S^2x, x, Sx) = 0.$$

It follows that $N\int_\Omega H(x,Sx,S^2x,...S^{N-1}x)d\mu=0$.

For the reverse implication, assume $\int_\Omega H(x,S_1x,S_2x,...,S_{N-1}x)d\mu(x)=0$ for every N-cyclically antisymmetric Hamiltonian H. By using the identity with Hamiltonians $(H_i)_{i=1}^N$ of the form

$$H_i(x_1,x_2,...,x_N):=f(x_1)-f(x_i)$$

where f is any continuous function on Ω, one gets that S_i is measure preserving for each $i=1,...,N-1$.

Now take for each fixed $i=1,...,N$, the Hamiltonian

$$H_i(x_1,x_2,...,x_N):=|x_i-S_1^ix_N|-|S_1^ix_1-x_{i+1}|-|x_{i+1}-S_1^ix_1|+|S_1^ix_2-x_{i+2}|.$$

Note that $H_i\in\mathcal{H}$ for each i, since it is of the form $H_i(x_1,...,x_N)=f(x_1,x_i,x_{i+1},x_N)-f(x_2,x_{i+1},x_{i+2},x_1)$. Now apply the idendity for each H_i to obtain,

$$0=\int_\Omega H_i(x,S_1x,S_2x,...,S_{N-1}x)d\mu(x)=0=\int_\Omega|S_{i-1}x-S_1^iS_{N-1}|\,d\mu+\int_\Omega|S_1^iS_1x-S_{i+1}x|\,d\mu=0.$$

It follows that $S_{i+1}=S_1^{i+1}$ and $S_{i-1}x=S_1^iS_{N-1}$ for each $i=1,...,N$. The latter applied to $i=1$, yields $x=S_1S_{N-1}x=S_1S_1^{N-1}x=S_1^Nx$, and we are done.

3 Regularization of N-sub-antisymmetric functions

Let Ω be a bounded domain in $\mathbb{R}^d$, and consider the class

$$\mathcal{H}_N^-(\Omega):=\{H\in C(\bar\Omega^N);\ \sum_{i=0}^{N-1}H(\sigma^i(\mathbf{x}))\le 0\ \text{ for all}\,\mathbf{x}\in\Omega^N\}. \tag{25}$$

For each $H\in\mathcal{H}_N^-(\Omega)$, we associate the following functional on $\Omega\times(\mathbb{R}^d)^{N-1}$,

$$L_H(x,p_1,...,p_{N-1})=\sup\left\{\sum_{i=1}^{N-1}\langle p_i,y_i\rangle-H(x,y_1,...,y_{N-1});y_i\in\Omega\right\}. \tag{26}$$

Denote by

$$\mathcal{L}_-(N)=\{L_H;H\in\mathcal{H}_N^-(\Omega)\}.$$

Our plan is to show that one can associate to H,

- a globally Lipschitz-continuous function $H^1_{reg}\in\mathcal{L}_-(N)$ that is concave in the first variable, convex in the last $(N-1)$ variables such that $L_{H^1_{reg}}\le L_H$.
- a globally Lipschitz-continuous function $H^2_{reg}\in\mathcal{L}(N)$ such that $H^2_{reg}\ge H^1_{reg}$ and hence $L_{H^2_{reg}}\le L_{H^1_{reg}}\le L_H$.

Suppose that Ω is contained in a ball B_R centered at the origin with radius $R>0$ in $\mathbb{R}^d$, we shall define *"an $(\bar\Omega\times B_R)$ restricted Legendre transform"* of L_H as

$$L_H^*(p_1,...,p_{N-1},,x)=\sup_{q\in\bar\Omega,y_i\in B_R}\left\{\langle q,x\rangle+\sum_{i=1}^{N-1}\langle p_i,y_i\rangle-L_H(q,y_1,y_2,...,y_{N-1})\right\}.$$

Similarly, we define on $\mathbb{R}^d\times(\mathbb{R}^d)^{N-1}$,

$$L_H^{**}(x,p_1,...,p_{N-1})=\sup_{p\in\bar\Omega,x_i\in B_R}\left\{\langle x,p\rangle+\sum_{i=1}^{N-1}\langle p_i,x_i\rangle-L_H^*(x_1,...,x_{N-1},p)\right\}. \tag{27}$$

For any function $L : \mathbb{R}^d \times (\mathbb{R}^d)^{N-1} \to \mathbb{R}$, we shall define its *"$B_R$-Hamiltonian"* by

$$H_L(x, y_1, ..., y_{N-1}) = \sup_{p_i \in B_R} \left\{ \sum_{i=1}^{N-1} \langle p_i, y_i \rangle - L(x, p_1, ..., p_{N-1}) \right\}. \tag{28}$$

Finally, for each $H \in \mathcal{H}_N^-(\Omega)$, we define the following two regularizations of H by

$$H^1_{reg}(\mathbf{x}) = H_{L_H^{**}}(\mathbf{x}), \tag{29}$$

and

$$H^2_{reg}(\mathbf{x}) = \frac{(N-1)H^1_{reg}(\mathbf{x}) - \sum_{i=1}^{N-1} H^1_{reg}(\sigma^i(\mathbf{x}))}{N}. \tag{30}$$

We list some of the properties of H^1_{reg}, H^2_{reg}, $L^1_{H_{reg}}$ and $L^1_{H_{reg}}$.

Proposition 3.1 *If $H \in \mathcal{H}_N^-(\Omega)$, then the following statements hold:*

1. *H^1_{reg} is a concave-convex on $\mathbb{R}^d \times \mathbb{R}^{d(N-1)}$ whose restriction to $\bar{\Omega}^N$ belong to $\mathcal{H}_N^-(\Omega)$.*
2. *H^2_{reg} belongs to $\mathcal{H}_N(\Omega)$, and $H^2_{reg} \geq H^1_{reg}$ on $\bar{\Omega}^N$.*
3. *$L_{H^1_{reg}}$ is convex and continuous in all variables and $L_{H^2_{reg}} \leq L_{H^1_{reg}} \leq L_H$ on $\bar{\Omega} \times (B_R)^{N-1}$.*
4. *$|L_{H^1_{reg}}(x, p_1, ..., p_{N-1})| \leq R\|x\| + R \sum_{i=1}^{N-1} \|p_i\| + (2N+1)R^2$ for all x and all $(p_i)_{i=1}^{N-1}$ in $\mathbb{R}^d$.*
5. *$|H^1_{reg}(x, y_1, ..., y_{N-1})| \leq R\|x\| + R \sum_{i=1}^{N-1} \|y_i\| + 2NR^2$ for all x and all $(y_i)_{i=1}^{N-1}$ in $\mathbb{R}^d$.*
6. *$L_{H^2_{reg}}$ and H^2_{reg} are both Lipschitz continuous with Lipschitz constants less than $4NR$.*

The proof will require several lemmas.

Lemma 3.1 *With the above notation, we have the following properties:*

1. *$L_H^{**}(x, p_1, ..., p_{N-1}) \leq L_H(x, p_1, ..., p_{N-1})$ for $x \in \bar{\Omega}$ and $p_i \in \mathbb{R}^d$ for $i = 1, ..., N-1$.*
2. *If $H^1_{\rm reg}$ denotes $H_{L_H^{**}}$, then $H^1_{\rm reg}$ is concave in the first variable and convex in the last $(N-1)$ variables.*
3. *$L_{H^1_{\rm reg}}$ is jointly convex in all variables.*

Proof. 1) For $x \in \bar{\Omega}$ and $p_i \in \mathbb{R}^d$, $i = 1, ..., N-1$, we have we have

$$\begin{aligned}
L_H^{**}(x, p_1, ..., p_{N-1}) &= \sup_{q \in \bar{\Omega}, r_i \in B_R} \left\{ \langle x, q \rangle + \sum_{i=1}^{N-1} \langle p_i, r_i \rangle - L_H^*(r_1, ..., r_{N-1}, q) \right\} \\
&= \sup_{q \in \bar{\Omega}, r_i \in B_R} \left\{ \langle x, q \rangle + \sum_{i=1}^{N-1} \langle p_i, r_i \rangle - \sup_{y \in \bar{\Omega}, y_i \in B_R} \{ \langle y, q \rangle + \sum_{i=1}^{N-1} \langle r_i, y_i \rangle - L_H(y, y_1, ..., y_{N-1}) \} \right\} \\
&= \sup_{q \in \bar{\Omega}, r_i \in B_R} \inf_{y \in \bar{\Omega}, y_i \in B_R} \left\{ \langle x, q \rangle + \sum_{i=1}^{N-1} \langle p_i, r_i \rangle - \langle y, q \rangle - \sum_{i=1}^{N-1} \langle r_i, y_i \rangle + L_H(y, y_1, ..., y_{N-1}) \right\} \\
&= \sup_{q \in \bar{\Omega}, r_i \in B_R} \inf_{y \in \bar{\Omega}, y_i \in B_R} \left\{ \langle q, x - y \rangle + \sum_{i=1}^{N-1} \langle p_i - y_i, r_i \rangle + L_H(y, y_1, ..., y_{N-1}) \right\} \\
&= \sup_{q \in \bar{\Omega}, r_i \in B_R} \inf_{y \in \bar{\Omega}, y_i \in B_R} \left\{ \langle q, x - y \rangle + \sum_{i=1}^{N-1} \langle p_i - y_i, r_i \rangle + \sup_{t_i \in \Omega} \{ \sum_{i=1}^{N-1} \langle t_i, y_i \rangle - H(y, t_1, ..., t_{N-1}) \} \right\} \\
&= \sup_{q \in \bar{\Omega}, r_i \in B_R} \inf_{y \in \bar{\Omega}, y_i \in B_R} \sup_{t_i \in \Omega} \left\{ \langle q, x - y \rangle + \sum_{i=1}^{N-1} \langle p_i - y_i, r_i \rangle + \sum_{i=1}^{N-1} \langle t_i, y_i \rangle - H(y, t_1, ..., t_{N-1}) \right\} \\
&= \inf_{y \in \bar{\Omega}, y_i \in B_R} \sup_{q \in \bar{\Omega}, r_i \in B_R} \sup_{t_i \in \Omega} \left\{ \langle q, x - y \rangle + \sum_{i=1}^{N-1} \langle p_i - y_i, r_i \rangle + \sum_{i=1}^{N-1} \langle t_i, y_i \rangle - H(y, t_1, ..., t_{N-1}) \right\}.
\end{aligned}$$

By taking $y=x$ and $y_i=p_i$, we readily get that $L_H^{**}(x,p_1,...,p_{N-1})\le L_H(x,p_1,...,p_{N-1})$.

For 2) note first that by definition

$$H_{L^{**}}(x,y_1,...,y_{N-1})=\sup_{p_i\in B_R}\left\{\sum_{i=1}^{N-1}\langle p_i,y_i\rangle - L_H^{**}(x,p_1,...,p_{N-1})\right\},$$

and therefore for all $x\in\mathbb{R}^d$, the function $(y_1,...,y_{N-1})\to H_{L^{**}}(x,y_1,...,y_{N-1})$ is convex. We shall show that for all $(y_1,...,y_{N-1})\in(\mathbb{R}^d)^{N-1}$, the function $x\to H_{L^{**}}(x,y_1,...,y_{N-1})$ is concave. In fact we show that

$$x\to -H_{L^{**}}(x,y_1,...,y_{N-1})=\inf_{p_i\in B_R}\{L_H^{**}(x,p_1,...,p_{N-1})-\sum_{i=1}^{N-1}\langle p_i,y_i\rangle\}$$

is convex. Indeed, consider $\lambda\in(0,1)$ and elements $x_1,x_2\in\mathbb{R}^d$, then for any a,b such that

$$a>-H_{L^{**}}(x_1,y_1,...,y_{N-1}) \text{ and } b>-H_{L^{**}}(x_2,y_1,...,y_{N-1}),$$

we can find $(r_i)_{i=1}^{N-1}$ and $(q_i)_{i=1}^{N-1}$ in $(\mathbb{R}^d)^{N-1}$ such that

$$-H_{L^{**}}(x_1,y_1,...,y_{N-1})\le L_H^{**}(x_1,r_1,...,r_{N-1})-\sum_{i=1}^{N-1}\langle r_i,y_i\rangle\le a,$$

and

$$-H_{L^{**}}(x_2,y_1,...,y_{N-1})\le L_H^{**}(x_2,q_1,...,q_{N-1})-\sum_{i=1}^{N-1}\langle q_i,y_i\rangle\le b.$$

Use the convexity of the ball B_R and the convexity of the function L_H^{**} in both variables to write

$$\begin{aligned}
-H_{L_H^{**}}(\lambda x_1+(1-\lambda)x_2,y_1,...,y_{N-1}) &= \inf_{p_i\in B_R}\{L_H^{**}(\lambda x_1+(1-\lambda)x_2,p_1,...,p_{N-1})-\sum_{i=1}^{N-1}\langle p_i,y_i\rangle\}\\
&\le L_H^{**}(\lambda x_1+(1-\lambda)x_2,\lambda r_1+(1-\lambda)q_1,...,\lambda r_{N-1}+(1-\lambda)q_{N-1}))\\
&\quad -\sum_{i=1}^{N-1}\langle\lambda r_i+(1-\lambda)q_i,y_i\rangle\\
&\le \lambda\big(L_H^{**}(x_1,r_1,...,r_{N-1})-\sum_{i=1}^{N-1}\langle r_i,y_i\rangle\big)\\
&\quad +(1-\lambda)\big(L_H^{**}(x_2,q_1,...,q_{N-1})-\sum_{i=1}^{N-1}\langle q_i,y_i\rangle\big)\}\\
&\le \lambda a+(1-\lambda)b,
\end{aligned}$$

which establishes the concavity of $x\to H_{L_H^{**}}(x,y_1,...,y_{N-1})$. It then follows that $L_{H^1_{reg}}=L_{H_{L_H^{**}}}$ is convex in all variables that proves part 3).

Lemma 3.2 *If $H\in\mathcal{H}_N^-(\Omega)$, then $H^1_{\rm reg}\in\mathcal{H}_N^-(\Omega)$.*

Proof. Let $i,j=1,2,..,N$. We first show that

$$\sum_{i=1}^{N}\Big\{\sum_{j=1,j\neq i}^{N}\langle p_j^i,x_j\rangle - L_H^{**}(R^{i-1}(p_1^i,...,p_{i-1}^i,x_i,p_{i+1}^i,...,p_N^i))\Big\}\le 0, \tag{31}$$

for all $x_i\in\Omega$ and $p_j^i\in\mathbb{R}^d$.

We have

$$L_H(\sigma^{i-1}(p^i_1,...,p^i_{i-1},x_i,p^i_{i+1},...,p^i_N)) = \sup\left\{\sum_{j=1,j\neq i}^{N}\langle p^i_j,y_j\rangle - H(\sigma^{i-1}(y_1,...,y_{i-1},x_i,y_{i+1},...,y_N)); y_j\in\Omega\right\}$$
$$\geq \sum_{j=1,j\neq i}^{N}\langle p^i_j,x_j\rangle - H(\sigma^{i-1}(x_1,x_2,...,x_n)).$$

Taking summation over i implies that

$$\sum_{i=1}^{N} L_H(\sigma^{i-1}(p^i_1,...,p^i_{i-1},x_i,p^i_{i+1},...,p^i_N)) \geq \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\langle p^i_j,x_j\rangle - \sum_{i=1}^{N} H(\sigma^{i-1}(x_1,x_2,...,x_n))$$

Since $\sum_{i=1}^N H(\sigma^{i-1}(x_1,x_2,...,x_n))\leq 0$, we obtain

$$\sum_{i=1}^{N} L_H(\sigma^{i-1}(p^i_1,...,p^i_{i-1},x_i,p^i_{i+1},...,p^i_N)) \geq \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\langle p^i_j,x_j\rangle.$$

It follows from the definition of L^{**}_H that

$$\sum_{i=1}^{N} L^{**}_H(\sigma^{i-1}(p^i_1,...,p^i_{i-1},x_i,p^i_{i+1},...,p^i_N)) \geq \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\langle p^i_j,x_j\rangle.$$

By moving the left hand side expression to the the other side, we have

$$0\geq \sum_{i=1}^{N}\Big\{\sum_{j=1,j\neq i}^{N}\langle p^i_j,x_j\rangle - L^{**}_H(\sigma^{i-1}(p^i_1,...,p^i_{i-1},x_i,p^i_{i+1},...,p^i_N))\Big\}.$$

Taking sup over all $p^j_i\in B_R$ we obtain $\sum_{i=1}^N H_{L^{**}_H}(\sigma^{i-1}(x_1,x_2,...,x_n))\leq 0$ and we are done. □

We now recall the following standard elementary result.

Lemma 3.3 *Let D be an open set in $\mathbb{R}^m$ such that $\bar{D}\subset\tilde{B}_R$ where $\tilde{B}_R$ is ball with radious R centered at the origin in $\mathbb{R}^m$. Let $f:\mathbb{R}^m\to\mathbb{R}$ and define $\tilde{f}:\mathbb{R}^m\to\mathbb{R}$ by*

$$\tilde{f}(y)=\sup_{z\in D}\{\langle y,z\rangle - f(z)\}.$$

If $f\in L^\infty(D)$, then $\tilde{f}$ is a convex Lipschitz function and

$$|\tilde{f}(y_1)-\tilde{f}(y_2)|\leq R\|y_1-y_2\| \text{ for all } y_1,y_2\in\mathbb{R}^m.$$

Lemma 3.4 *If $H\in\mathcal{H}^-_N(\Omega)$, then the following statements hold:*

1. *$|L^{**}_H(x,p_1,...,p_{N-1})|\leq R\|x\|+R\sum_{i=1}^{N-1}\|p_i\|+(2N-1)R^2$ for all x and $(p_i)_{i=1}^{N-1}$ in $\mathbb{R}^d$.*
2. *$|H_{L^{**}_H}(x,y_1,...,y_{N-1})|\leq R\|x\|+R\sum_{i=1}^{N-1}\|y_i\|+2NR^2$ for all x and $(y_i)_{i=1}^{N-1}$ in $\mathbb{R}^d$.*
3. *L^{**}_H and $H_{L^{**}_H}$ are Lipschitz continuous with Lipschitz constants $Lip(H_{L^{**}_H}), Lip(L^{**}_H)\leq NR$.*

Proof. Since H is N-sub-antisymmetric, we have $H(x,...,x)\leq 0$, hence

$$L_H(x,p_1,...,p_{N-1})\geq\sum_{i=1}^{N-1}\langle p_i,x\rangle \quad\text{on } \bar{\Omega}\times(\mathbb{R}^d)^{N-1}.$$

This together with the fact that $\bar{\Omega} \subset B_R$ imply that

$$\begin{aligned} L_H^*(p_1,...,p_{N-1},x) &= \sup_{q\in\bar{\Omega}, y_i\in B_R} \left\{ \langle q,x\rangle + \sum_{i=1}^{N-1} \langle p_i, y_i\rangle - L_H(q,y_1,y_2,...,y_{N-1}) \right\}. \\ &\leq \sup_{q\in\bar{\Omega}, y_i\in B_R} \left\{ \langle q,x\rangle + \sum_{i=1}^{N-1} \langle p_i, y_i\rangle - \sum_{i=1}^{N-1} \langle q, y_i\rangle \right\}. \\ &\leq R\|x\| + R\sum_{i=1}^{N-1} \|p_i\| + (N-1)R^2. \end{aligned}$$

With a similar argument we obtain that $L_H^{**}(x,p_1,...,p_{N-1}) \leq R\|x\| + R\sum_{i=1}^{N-1}\|p_i\| + (N-1)R^2$. We also have

$$\begin{aligned} L_H^{**}(x,p_1,...,p_{N-1}) &= \sup_{p\in\bar{\Omega}, x_i\in B_R} \left\{ \langle x,p\rangle + \sum_{i=1}^{N-1} \langle p_i, x_i\rangle - L_H^*(x_1,...,x_{N-1},p) \right\} \\ &\geq \langle x,p\rangle + \sum_{i=1}^{N-1} \langle p_i, x_i\rangle - L_H^*(x_1,...,x_{N-1},p) \\ &\geq -R\|x\| - R\sum_{i=1}^{N-1}\|p_i\| - R\|p\| - R\sum_{i=1}^{N-1}\|x_i\| - (N-1)R^2 \\ &\geq -R\|x\| - R\sum_{i=1}^{N-1}\|p_i\| - (2N-1)R^2. \end{aligned}$$

Therefore $|L_H^{**}(x,p_1,...,p_{N-1})| \leq R\|x\| + R\sum_{i=1}^{N-1}\|p_i\| + (2N-1)R^2$. The estimate for $H_{L_H^{**}}$ can be easily deduced from its definition together with the estimate on L_H^{**}. This completes the proof of part (1).

For (2) set $D = \Omega \times \Pi_{i=1}^{N-1} B_R$, then $D \subset \tilde{B}_{NR}$ where $\tilde{B}_{NR}$ is a ball with radius NR in $\mathbb{R}^{dN}$. Now assuming $f = L_H^*$ in Lemma 3.3, we have that $\tilde{f} = L_H^{**}$. Therefore L_H^{**} is Lipschitz in $(\mathbb{R}^d)^N$ with $Lip(L_H^{**}) \leq NR$. To prove that $H_{L_H^{**}}$ is Lipschitz continuous, we first fix $y \in \mathbb{R}^d$ and define $f_y : (\mathbb{R}^d)^{N-1} \to \mathbb{R}$ by

$$f_y(p_1,...,p_{N-1}) = L_H^{**}(y,p_1,...p_{N-1}).$$

Assuming $D = B_R \subset \mathbb{R}^N$ in Proposition 3.3, we obtain that the map

$$(x_1,...,x_{N-1}) \to \tilde{f}_y(x_1,...,x_{N-1}) = H_{L_H^{**}}(y,x_1,...,x_{N-1})$$

is Lipschitz and

$$|H_{L_H^{**}}(y,x_1,...,x_{N-1}) - H_{L_H^{**}}(y,z_1,...,z_{N-1})| \leq R\sum_{i=1}^{N-1}\|x_i - z_i\| \tag{32}$$

for all $(x_i),(z_i) \in (\mathbb{R}^d)^{N-1}$. Noticing that the Lipschitz constant R is independent of y, the above inequality holds for all $(x_i),(z_i) \in (\mathbb{R}^d)^{N-1}$ and $y \in \mathbb{R}^d$. To prove $H_{L_H^{**}}(y,x_1,...,x_{N-1})$ is Lipschitz with respect to the first variable y, let $r > 0$ and $y_1, y_2 \in \mathbb{R}^d$. Let $p_1,...,p_{N-1}$ and $q_1,...,q_{N-1}$ be such that

$$\sum_{i=1}^{N-1}\langle x_i,q_i\rangle - L_H^{**}(y_1,q_1,...,q_{N-1}) \leq H_{L_H^{**}}(y_1,x_1,...,x_{N-1}) \leq \sum_{i=1}^{N-1}\langle x_i,p_i\rangle - L_H^{**}(y_1,p_1,...,p_{N-1}) + r,$$

and

$$\sum_{i=1}^{N-1}\langle x_i,p_i\rangle - L_H^{**}(y_2,p_1,...,p_{N-1}) \leq H_{L_H^{**}}(y_2,x_1,...,x_{N-1}) \leq \sum_{i=1}^{N-1}\langle x_i,q_i\rangle - L_H^{**}(y_2,q_1,...,q_{N-1}) + r,$$

It follows that

$$\begin{array}{rcl} L_H^{**}(y_2,q_1,...,q_{N-1}) - L_H^{**}(y_1,q_1,...,q_{N-1}) - r & \leq & H_{L_H^{**}}(y_1,x_1,...,x_{N-1}) - H_{L_H^{**}}(y_2,x_1,...,x_{N-1}) \\ & \leq & L_H^{**}(y_2,p_1,...,p_{N-1}) - L_H^{**}(y_1,p_1,...,p_{N-1}) + r. \end{array}$$

Since L_H^{**} is Lipschitz,

$$-NR\|y_1-y_2\| - r \leq H_{L_H^{**}}(y_1,x_1,...,x_{N-1}) - H_{L_H^{**}}(y_2,x_1,...,x_{N-1}) \leq NR\|y_1-y_2\| + r.$$

Since $r>0$ is arbitrary we obtain

$$-NR\|y_1-y_2\| \leq H_{L_H^{**}}(y_1,x_1,...,x_{N-1}) - H_{L_H^{**}}(y_2,x_1,...,x_{N-1}) \leq NR\|y_1-y_2\|.$$

This together with (32) prove that $H_{L_H^{**}}$ is Lipschitz continuous and that $Lip(H_{L_H^{**}}) \leq NR$. □

Proof of Proposition 3.1. 1) By Lemma 3.2, we have that $H^1_{reg} := H_{L_H^{**}}$ is a concave-convex Hamiltonian on $\mathbb{R}^d \times (\mathbb{R}^d)^{N-1}$ whose restriction to $\bar{\Omega}^N$ is N-sub-antisymmetric, hence belong to $\mathcal{H}_N^-(\Omega)$.

2) To show that H^2_{reg} is N-antisymmetric note that

$$NH^2_{reg}(\mathbf{x}) = (N-1)H^1_{reg}(\mathbf{x}) - \sum_{i=1}^{N-1} H^1_{reg}(R^i(\mathbf{x})) = \sum_{i=1}^{N-1} \big[H^1_{reg}(\mathbf{x}) - H^1_{reg}(R^i(\mathbf{x}))\big]$$

and each of the terms $H^1_{reg}(\mathbf{x}) - H^1_{reg}(R^i(\mathbf{x}))$ is easily seen to be N-antisymmetric.

Now H^2_{reg} dominates H^1_{reg} since

$$N\big[H^2_{reg}(\mathbf{x}) - H^1_{reg}(\mathbf{x})\big] = -H^1_{reg}(\mathbf{x}) - \sum_{i=1}^{N-1} H^1_{reg}(R^i(\mathbf{x})) \geq 0,$$

since H^1_{reg} is N-sub-antisymmetric.

3) For $x \in \Omega$ and $p_1,...,p_{N-1} \in B_R$ we have

$$\begin{array}{rcl} L_{H^1_{reg}}(x,p_1,...,p_{N-1}) & = & \sup_{y_i\in\Omega} \Big\{ \sum_{i=1}^{N-1} \langle p_i, y_i\rangle - H_{L_H^{**}}(x,y_1,...,y_{N-1}) \Big\} \\ & = & \sup_{y_i\in\Omega} \Big\{ \sum_{i=1}^{N-1} \langle p_i, y_i\rangle - \sup_{q_i\in B_R} \{ \sum_{i=1}^{N-1} \langle q_i, y_i\rangle - L_H^{**}(x,q_1,...,q_{N-1})\} \Big\} \\ & = & \sup_{y_i\in\Omega} \inf_{q_i\in B_R} \Big\{ \sum_{i=1}^{N-1} \langle p_i, y_i\rangle - \sum_{i=1}^{N-1} \langle q_i, y_i\rangle + L_H^{**}(x,q_1,...,q_{N-1})\} \Big\} \\ & \leq & \inf_{q_i\in B_R} \sup_{y_i\in\Omega} \Big\{ \sum_{i=1}^{N-1} \langle p_i, y_i\rangle - \sum_{i=1}^{N-1} \langle q_i, y_i\rangle + L_H^{**}(x,q_1,...,q_{N-1})\} \Big\} \\ & \leq & \sup_{y_i\in\Omega} \Big\{ \sum_{i=1}^{N-1} \langle p_i, y_i\rangle - \sum_{i=1}^{N-1} \langle p_i, y_i\rangle + L_H^{**}(x,p_1,...,p_{N-1})\} \Big\} \\ & = & L_H^{**}(x,p_1,...,p_{N-1}) \end{array}$$

On the other hand by Lemma (3.1) we have $L_H^{**} \leq L_H$, and therefore $L_{H^1_{reg}} \leq L_H$. It also follows from part 2) that $L_{H^2_{reg}} \leq L_{H^1_{reg}}$. This completes the proof of part 3).

Parts 4), 5) and 6) are the subject of the preceding Lemmas. □

4 Proof of Theorem 1.1: Existence

We first show that the minimization problem (DK) has a solution. Let B_R be a ball such that $\bar{\Omega}$ and $u_i(\bar{\Omega}) \subset B_R$ for all $i = 1, ..., N-1$. Let $\{H_n\}$ be a sequence in $\mathcal{H}$ such that L_{H_n} is a minimizing sequence for (DK). Denoting $H_n^1 := (H_n)^1_{reg}$, we get from Proposition 3.1 that $L_{H_n^1} \leq L_{H_n}$ on $\bar{\Omega} \times B_R^{N-1}$ and therefore $L_{H_n^1}$ is also minimizing for (DK). It also follows from Proposition 3.1 that $L_{H_n^1}$ and H_n^1, are uniformly Lipschitz with $Lip(H_n^1), Lip(L_{H_n^1}) \leq NR$. Moreover,

$$|H_n^1(x, y_1, ..., y_{N-1})| \leq R\|x\| + R\sum_{i=1}^{N-1} \|y_i\| + 2NR^2 \text{ for all } x \text{ and } (y_i)_{i=1}^{N-1} \text{ in } \mathbb{R}^d,$$

and

$$|L_{H_n^1}(x, p_1, ..., p_{N-1})| \leq R\|x\| + R\sum_{i=1}^{N-1} \|p_i\| + (2N-1)R^2 \text{ for all } x, p_1, ..., p_{N-1} \text{ in } \mathbb{R}^d.$$

By Arzela-Ascoli's theorem, there exists two Lipschitz functions $\tilde{H}$ and $\tilde{L} : \mathbb{R}^d \times \mathbb{R}^{d(N-1)} \to \mathbb{R}$ such that H_n^1 converges to $\tilde{H}$ and L_n^1 converges to $\tilde{L}$ uniformly on every compact set of $\mathbb{R}^N \times ... \times \mathbb{R}^N$. This implies that $\tilde{H} \in \mathcal{H}_N^-(\Omega)$. Note that

$$L_{H_n^1}(x, p_1, ..., p_{N-1}) + H_n^1(x, y_1, ..., y_{N-1}) \geq \sum_{i=1}^{N-1} \langle y_i, p_i \rangle,$$

for all $x, p_1, ..., p_{N-1} \in \mathbb{R}^N$ and $y_1, ..., y_{N-1} \in \bar{\Omega}$, from which we have

$$\tilde{L}(x, p_1, ..., p_{N-1}) \geq \sum_{i=1}^{N-1} \langle y_i, p_i \rangle - \tilde{H}(x, y_1, ..., y_{N-1}),$$

for all $x, p_1, ..., p_{N-1} \in \mathbb{R}^N$ and $y_1, ..., y_{N-1} \in \bar{\Omega}$. This implies that $L_{\tilde{H}} \leq \tilde{L}$. Let $H^1_\infty = \tilde{H}^1_{reg}$ and $H^2_\infty = \tilde{H}^2_{reg}$ be the regularizations of $\tilde{H}$ defined in the previous section. Set $L^i_\infty = L_{H^i_\infty}$ for $i = 1, 2$. It follows from Proposition 3.1 that $L_{H^2_\infty} \leq L_{H^1_\infty} \leq L_{\tilde{H}}$ on $\bar{\Omega} \times B_R^{N-1}$, from which we have

$$\begin{aligned} DK &= \int_\Omega L_{\tilde{H}}(x, u_1(x), ..., u_{N-1}(x))\, d\mu(x) \\ &= \int_\Omega L^2_\infty(x, u_1(x), ..., u_{N-1}(x))\, d\mu(x) \\ &= \int_\Omega L^1_\infty(x, u_1(x), ..., u_{N-1}(x))\, d\mu(x). \end{aligned}$$

□

For the rest of the proof, we shall need the following two technical lemmas. The first one relates L_H^* to the standard Legendre transform of H (extended beyond Ω^N to the whole of $\mathbb{R}^{dN}$.)

Lemma 4.1 *Let $H_\infty = H^1_\infty$ be the concave-convex Hamiltonian obtained above and $L_\infty = L^1_\infty$. For each $x \in \bar{\Omega}$, define $f_x : (\mathbb{R}^d)^{N-1} \to \mathbb{R}$ by*

$$f_x(y_1, ..., y_{N-1}) := H_\infty(x, y_1, ..., y_{N-1}).$$

We also define $\tilde{f}_x : (\mathbb{R}^d)^{N-1} \to \mathbb{R} \cup \{+\infty\}$ by

$$\tilde{f}_x(y_1, ..., y_{N-1}) := f_x(y_1, ..., y_{N-1}) \text{ if } y_1, ..., y_{N-1} \in \bar{\Omega}^{N-1} \text{ and } +\infty \text{ otherwise.}$$

Let $(\tilde{f}_x)^*$ be the standard Fenchel dual of $\tilde{f}_x$ on $(\mathbb{R}^d)^{N-1}$ in such a way that $(\tilde{f}_x)^{***} = (\tilde{f}_x)^*$ on $(\mathbb{R}^d)^{N-1}$. We then have,

$$f_x = (\tilde{f}_x)^{**} = \tilde{f}_x \text{ on } \bar{\Omega}^{N-1} \tag{33}$$

and

$$\begin{aligned} L_\infty(x, p_1, ..., p_{N-1}) &= \sup_{(z_i) \in \bar{\Omega}^{N-1}} \{\sum_{i=1}^{N-1} \langle z_i, p_i \rangle - (\tilde{f}_x)^{**}(z_1, ..., z_{N-1})\} \\ &= \sup_{(z_i) \in (\mathbb{R}^d)^{N-1}} \{\sum_{i=1}^{N-1} \langle z_i, p_i \rangle - (\tilde{f}_x)^{**}(z_1, ..., z_{N-1})\}. \end{aligned} \tag{34}$$

Proof. Since $(\tilde{f}_x)^{**}$ is the largest convex function below $\tilde{f}_x$ we have and $f_x \le (\tilde{f}_x)^{**} \le \tilde{f}_x$, from which we obtain $f_x = (\tilde{f}_x)^{**} = \tilde{f}_x$ on $\bar{\Omega}^{N-1}$.

For (34), we first deduce from (33) that

$$\begin{aligned}
(\tilde{f}_x)^*(y_1,...,y_{N-1}) &= (\tilde{f}_x)^{***}(y_1,...,y_{N-1}) \\
&= \sup_{z\in\mathbb{R}^{d(N-1}} \{\sum_{i=1}^{N-1} \langle z_i, y_i\rangle - (\tilde{f}_x)^{**}(z_1,...,z_{N-1})\} \\
&\ge \sup_{z\in B_R^{N-1}} \{\sum_{i=1}^{N-1} \langle z_i, y_i\rangle - (\tilde{f}_x)^{**}(z_1,...,z_{N-1})\} \\
&\ge \sup_{z\in\Omega^{N-1}} \{\sum_{i=1}^{N-1} \langle z_i, y_i\rangle - (\tilde{f}_x)^{**}(z_1,...,z_{N-1})\} \\
&= \sup_{z\in\Omega^{N-1}} \{\sum_{i=1}^{N-1} \langle z_i, y_i\rangle - f_x(z_1,...,z_{N-1})\} \\
&= \sup_{z\in\Omega^{N-1}} \{\sum_{i=1}^{N-1} \langle z_i, y_i\rangle - \tilde{f}_x(z_1,...,z_{N-1})\} \\
&= (\tilde{f}_x)^*(y_1,...,y_{N-1}),
\end{aligned}$$

from which we have the desired result. □

Fix now H_∞ as above and let $H \in C(\bar{\Omega}^N)$. For each $\lambda > 0$ and $r \in (-1,1)$, we associated the following three functionals.

$$\begin{aligned}
L_{r,\lambda}(x,p_1,...,p_{N-1}) &:= \sup_{(z_i)\in\bar{\Omega}^{N-1}} \Big\{ \sum_{i=1}^{N-1} \langle z_i, p_i\rangle - (\tilde{f}_x)^{**}(z_1,...,z_{N-1}) - \frac{\lambda}{2}[\sum_{i=1}^{N-1} \|z_i\|^2 - (N-1)\|x\|^2] \\
&\qquad + rH(x,z_1,...,z_{N-1})\} \\
L_\lambda(x,p_1,...,p_{N-1}) &:= \sup_{(z_i)\in\mathbb{R}^{d(N-1)}} \Big\{ \sum_{i=1}^{N-1} \langle z_i, p_i\rangle - (\tilde{f}_x)^{**}(z_1,...,z_{N-1}) - \frac{\lambda}{2}[\sum_{i=1}^{N-1} \|z_i\|^2 - (N-1)\|x\|^2] \Big\} \\
L_r(x,p_1,...,p_{N-1}) &:= \sup_{(z_i)\in\bar{\Omega}^{N-1}} \Big\{ \sum_{i=1}^{N-1} \langle z_i, p_i\rangle - H_\infty(x,z_1,...,z_{N-1}) + rH(x,z_1,...,z_{N-1}) \Big\}.
\end{aligned}$$

Lemma 4.2 *Let $H \in C(\bar{\Omega}^N)$ be such that $H_\infty - rH \in \mathcal{H}_N^-(\Omega)$ for all $r \in (-1,1)$. Then the following hold:*

1. *For every $(x,p_1,...,p_{N-1}) \in \mathbb{R}^d \times \mathbb{R}^{d(N-1)}$, we have*

$$\lim_{\lambda\to 0^+} L_\lambda(x,p_1,...,p_{N-1}) = L_\infty(x,p_1,...,p_{N-1}) \text{ and } \lim_{\lambda\to 0^+} L_{r,\lambda}(x,p_1,...,p_{N-1}) = L_r(x,p_1,...,p_{N-1}).$$

2. *For all $x \in \mathbb{R}^d$, the function $(p_1,...,p_{N-1}) \to L_\lambda(x,p_1,...,p_{N-1})$ is differentiable.*
3. *For every $(x,p_1,...,p_{N-1}) \in \mathbb{R}^d \times \mathbb{R}^{d(N-1)}$, we have*

$$\lim_{r\to 0} \frac{L_{r,\lambda}(x,p_1,...,p_{N-1}) - L_\lambda(x,p_1,...,p_{N-1})}{r} = H(\nabla_{2,...,N} L_\lambda(x,p_1,...,p_{N-1}), x).$$

Proof. Yosida's regularization of convex functions and Lemma 4.1 yield that

$$\begin{aligned}
\lim_{\lambda\to 0^+} L_{r,\lambda}(x,p_1,...,p_{N-1}) &= \sup_{(z_i)\in\bar{\Omega}^{N-1}} \{\sum_{i=1}^{N-1} \langle z_i, p_i\rangle - (\tilde{f}_x)^{**}(z_1,...,z_{N-1}) - rH(x,z_1,...,z_{N-1})\} \\
&= \sup_{(z_i)\in\bar{\Omega}^{N-1}} \{\sum_{i=1}^{N-1} \langle z_i, p_i\rangle - H_\infty(x,z_1,...,z_{N-1}) - rH(x,z_1,...,z_{N-1})\} \\
&= L_r(x,p_1,...,p_{N-1}).
\end{aligned}$$

We also have

$$\lim_{\lambda\to 0} L_\lambda(x,p_1,...,p_{N-1}) = \sup_{(z_i)\in\mathbb{R}^{d(N-1)}} \{\sum_{i=1}^{N-1}\langle z_i,p_i\rangle - (\tilde{f}_x)^{**}(z_1,...,z_{N-1})\},$$

which, together with Lemma 4.1, yield that $\lim_{\lambda\to 0} L_\lambda(x,p_1,...,p_{N-1}) = L_\infty(x,p_1,...,p_{N-1})$.

(2) follows from the fact that the Yosida regularization of convex functions are differentiable.

(3) We let $z_{(r,\lambda,i)} \in \bar{\Omega}$ and $z'_{(r,\lambda,i)} \in \mathbb{R}^d$ be such that

$$\begin{aligned} L_{r,\lambda}(x,p_1,...,p_{N-1}) &\le \sum_{i=1}^{N-1}\langle z_{(r,\lambda,i)},p_i\rangle - (\tilde{f}_x)^{**}(z_{(r,\lambda,1)},...,z_{(r,\lambda,N-1)}) - \frac{\lambda}{2}\sum_{i=1}^{N-1}\|z_{(r,\lambda,i)}\|^2 \\ &\quad +\lambda\frac{(N-1)\|x\|^2}{2} + rH(x,z_{(r,\lambda,1)},...,z_{(r,\lambda,N-1)}) + r^2, \\ L_\lambda(x,p_1,...,p_{N-1}) &\le \sum_{i=1}^{N-1}\langle z'_{\lambda,i},p_i\rangle - (\tilde{f}_x)^{**}(z'_{(r,\lambda,1)},...,z_{(r,\lambda,N-1)}) - \frac{\lambda}{2}\sum_{i=1}^{N-1}\|z'_{(r,\lambda,i)}\|^2 + \lambda\frac{(N-1)\|x\|^2}{2} + r^2. \end{aligned}$$

Therefore,

$$\begin{aligned} rH(x,z'_{(r,\lambda,1)},...,z'_{(r,\lambda,N-1)}) - r^2 &\le L_{r,\lambda}(x,p_1,...,p_{N-1}) - L_\lambda(x,p_1,...,p_{N-1}) \\ &\le rH(x,z_{(r,\lambda,1)},...,z_{(r,\lambda,N-1)}) + r^2. \end{aligned} \tag{35}$$

By the definition of L_λ, we have $\sup_{r\in[-1,1]}\|z'_{r,\lambda,i}\| < \infty$. Suppose now that, up to a subsequence, $z_{r,\lambda,i} \to z_i \in \bar{\Omega}$ and $z'_{r,\lambda,i} \to z'_{\lambda,i}$ as $r\to 0$. This together with the definition of $L_{r,\lambda}$ and L_λ imply that

$$\begin{aligned} L_\lambda(x,p_1,...,p_{N-1}) &= \sum_{i=1}^{N-1}\langle z_{(\lambda,i)},p_i\rangle - (\tilde{f}_x)^{**}(z_{(\lambda,1)},...,z_{(\lambda,N-1)}) - \frac{\lambda}{2}\sum_{i=1}^{N-1}\|z_{(\lambda,i)}\|^2 + \lambda(N-1)\frac{\|x\|^2}{2} \\ &= \sum_{i=1}^{N-1}\langle z'_{\lambda,i},p_i\rangle - (\tilde{f}_x)^{**}(z_{(\lambda,1)},...,z'_{(\lambda,N-1)}) - \frac{\lambda}{2}\sum_{i=1}^{N-1}\|z'_{(\lambda,i)}\|^2 + \lambda(N-1)\frac{\|x\|^2}{2}, \end{aligned}$$

from which we obtain that

$$z_{\lambda,i} = z'_{\lambda,i} = \nabla_i L_\lambda(x,p_1,...,p_{N-1}) \in \bar{\Omega}, \qquad i=2,...,N \tag{36}$$

Therefore, it follows from (35) that

$$\lim_{r\to 0}\frac{L_{r,\lambda}(x,p_1,...,p_{N-1}) - L_\lambda(x,p_1,...,p_{N-1})}{r} = H\big(\nabla_{2,...,N}L_\lambda(x,p_1,...,p_{N-1}),x\big).$$

□

End of the proof of Theorem 1.1: For each $\lambda>0$, $x\in\bar{\Omega}$ and $p\in\mathbb{R}^N$, we define

$$\bar{S}_{\lambda,i}(x,p_1,...,p_{N-1}) = \nabla_i L_\lambda(x,p_1,...,p_{N-1}) \quad i=2,...,N$$

It is easy to see that $\bar{S}_{\lambda,i}(x,p_1,...,p_{N-1}) \to \bar{S}_{0,i}(x,p_1,...,p_{N-1})$ where $\bar{S}_{0,i}(x,p_1,...,p_{N-1})$ is the unique element with minimal norm in $\partial_i L_\infty(x,p_1,...,p_{N-1})$. Set $S_{\lambda,i}(x) = \bar{S}_{\lambda,i}(x,u_1(x),...,u_{N-1}(x))$, and $S_i(x) = S_{0,i}(x,u_1(x),...,u_{N-1}(x))$. For each $r>0$, $\lambda\in[0,1]$ and $x\in\bar{\Omega}$, define

$$\eta_r(\lambda,x) = \frac{L_{r,\lambda}(x,u_1(x),...,u_{N-1}x) - L_\lambda(x,u_1(x),...,u_{N-1}(x))}{r}.$$

Note that the function $r\to L_{r,\lambda}(x,u_1(x),...,u_{N-1}(x))$ is a convex function because it is supremum of a family of linear functions. Thus, for fixed $(x,\lambda)\in\Omega\times[0,1]$, the function $r\to\eta_r(\lambda,x)$ is non-decreasing. Setting $\eta_0(\lambda,x)$ to be $H(x,S_{\lambda,1}(x),...,S_{\lambda,N-1}(x))$ for $\lambda>0$ and $\eta_0(0,x) = H(x,S_1(x),...,S_{N-1}(x))$, we have that both functions $\lambda\to\eta_r(\lambda,x)$ and $\lambda\to\eta_0(\lambda,x)$ are continuous. It follows from Dini's Theorem, that for a fixed x, $\eta_r(\lambda, x)$ converges uniformly to $\eta_0(\lambda, x)$ as $r \to 0$ with respect to $\lambda \in [0,1]$. Note also that thanks to (36) we have that $S_{\lambda,i}, S_i : \bar{\Omega} \to \bar{\Omega}$ and for all $x \in \Omega$.

$$(S_1x, ..., S_{N-1}x) \in \partial_{2,...,N} L_\infty(x, u_1(x), ..., u_{N-1}(x)). \tag{37}$$

We now show that

$$\textstyle\int_\Omega H(x, S_1x, ..., S_{N-1}x)\, d\mu(x) = 0 \text{ for all } H \in C(\bar{\Omega}^N) \text{ with } H_\infty - rH \in \mathcal{H}_N^-(\Omega),\ r \in (-1,1). \tag{38}$$

Indeed, since $|H(x, S_1x, ..., S_{N-1}x)| \le \|H\|_{L^\infty(\bar{\Omega}^N)}$, we get from Lebesgue's dominated convergence Theorem,

$$\lim_{\lambda\to 0} \int_\Omega H(x, S_{\lambda,1}(x), ..., S_{\lambda,N-1}(x))\, d\mu(x) = \int_\Omega H(x, S_1x, ..., S_{N-1}x)\, d\mu(x).$$

From (35) we have

$$\Big|\frac{L_{r,\lambda}(x, p_1, ..., p_{N-1}) - L_\lambda(x, p_1, ..., p_{N-1})}{r}\Big| \le \|H\|_{L^\infty(\bar{\Omega}^N)} + |r|,$$

from which follows that

$$\begin{aligned}
\int_\Omega H(x, S_1x, ..., S_{N-1}x)\, d\mu(x) &= \int_\Omega \lim_{\lambda\to 0}\lim_{r\to 0^+} \frac{L_{r,\lambda}(x, u_1(x), ..., u_{N-1}(x)) - L_\lambda(x, u_1(x), ..., u_{N-1}(x))}{r}\, d\mu(x)\\
&= \int_\Omega \lim_{\lambda\to 0}\lim_{r\to 0^+} \eta_r(\lambda, x)\, d\mu(x)\\
&= \int_\Omega \lim_{r\to 0^+}\lim_{\lambda\to 0} \eta_r(\lambda, x)\, d\mu(x) \qquad \text{(due to the uniform convergence)}\\
&= \int_\Omega \lim_{r\to 0^+} \eta_r(0, x)\, d\mu(x)\\
&= \lim_{r\to 0^+} \int_\Omega \eta_r(0, x)\, d\mu(x) \qquad \text{(due to the monotone convergence theorem)}\\
&= \lim_{r\to 0^+} \int_\Omega \frac{L_r(x, u_1(x), ..., u_{N-1}(x)) - L_\infty(x, u_1(x), ..., u_{N-1}(x))}{r}\, d\mu(x)\\
&\ge 0, \qquad \text{(in view of the optimality of } H_\infty \text{ compared to } H_\infty - rH).
\end{aligned}$$

In other words, we have $\int_\Omega H(x, S_1x, ..., S_{N-1}x)\, d\mu(x) \ge 0$. By the same argument considering $r \to 0^-$, one has $\int_\Omega H(x, S_1x, ..., S_{N-1}x)\, d\mu(x) \le 0$ and therefore the latter is indeed zero as desired.

Note now that (38) yields that both

$$\int_\Omega H_\infty(x, S_1x, ..., S_{N-1}x)\, d\mu(x) = 0, \tag{39}$$

and that

$$\textstyle\int_\Omega H(x, S_1x, ..., S_{N-1}x)\, d\mu(x) = 0 \text{ for all } H \in \mathcal{H}_N(\Omega). \tag{40}$$

It follows from Lemma 2.3 that S is measure preserving, that $S_i = S_1^i$ and that $S_1^N = I$. We shall now write S for S_1.

We now show that $DK = MK$. We already know that $MK \le DK$. To prove the equality, we use the fact that $(Sx, ..., S^{N-1}x) \in \partial_{2,...,N} L_\infty(x, u_1(x), ..., u_{N-1}(x))$ together with $(\tilde{f}_x)^{**}$ being the Fenchel dual of L with respect to the last $N-1$ variables and Lemma 4.1 to obtain that

$$(u_1(x), ..., u_{N-1}(x)) \in \partial(\tilde{f}_x)^{**}(Sx, ..., S^{N-1}x). \tag{41}$$

Since $\text{meas}(\partial\Omega) = 0$, the set $\cup_{i=1}^{N-1} S^{-i}(\partial\Omega)$ is negligible and for each $x \in \Omega \setminus \cup_{i=1}^{N-1} S^{-i}(\partial\Omega)$, one has

$$\partial(\tilde{f}_x)^{**}(Sx, ..., S^{N-1}x) = \partial_{2,...,N} H_\infty(x, Sx, ..., S^{N-1}x).$$

It follows that

$$(u_1(x),...,u_{N-1}(x)) \in \partial_{2,...,N} H_\infty(x, Sx, ..., S^{N-1}x) \qquad a.e. \quad x \in \Omega. \tag{42}$$

We finally get that

$$\begin{aligned}
DK &= \int_\Omega L_\infty(x, u_1(x), ..., u_{N-1}(x))\, d\mu(x) \\
&= \int_\Omega L_\infty(x, u_1(x), ..., u_{N-1}(x))\, d\mu(x) + \int_\Omega H_\infty(x, Sx, ..., S^{N-1}x)\, d\mu(x) \\
&= \int_\Omega L_\infty(x, u_1(x), ..., u_{N-1}(x))\, d\mu(x) + \int_\Omega (\tilde{f}_x)^{**}(Sx, ..., S^{N-1}x)\, d\mu(x) \\
&= \int_\Omega \sum_{i=1}^{N-1} \langle u_i(x), S^i(x)\rangle\, d\mu(x) \le MK.
\end{aligned}$$

If now $u_i \in W^{1,1}_{loc}(\Omega)$ for $i = 1, 2, ..., N-1$, or if S is a.e. differentiable, then by Theorem 7.1 of the Appendix, there exists a full measure subset Ω_0 of Ω that $\nabla_{2,...,N} H_\infty(x, Sx, ..., S^{N-1}x)$ exists for all $x \in \Omega_0$. It follows that

$$\big(u_1(x), ..., u_{N-1}(x)\big) = \nabla_{2,...,N} H_\infty(x, Sx, ..., S^{N-1}x) \quad \text{ for all } x \in \Omega_0.$$

5 Proof of Theorem 1.1: Uniqueness

We now deal with part (5) of Theorem 1.1. H_∞ will denote an optimal concave-convex N-sub-antisymmetric associated to the vector fields $u_1, ..., u_{N-1}$ via the above variational procedure.

Lemma 5.1 *Assume that the vector fields $u_1, ..., u_{N-1}$ from Ω to $\mathbb{R}^d$ are such that*

$$\big(u_1(x), ..., u_{N-1}(x)\big) \in \partial_{2,...,N} H_1(x, Sx, ..., S^{N-1}x) \quad a.e. \ x \in \Omega,$$

for some concave-convex N-sub-antisymmetric Hamiltonian H_1 and some N-involution S, then (H_1, S) is an "extremal pair", meaning that the infimum (DK) is attained at H_1 and the supremum (MK') is attained at S. Moreover, we have

$$\big(u_1(x), ..., u_{N-1}(x)\big) \in \partial_{2,...,N} H_\infty(x, Sx, ..., S^{N-1}x) \quad a.e. \ x \in \Omega,$$

where H_∞ is the optimal Hamiltonian constructed above.

Proof. Let L be the Fenchel-Legendre dual of H_1 with respect to the last $N-1$ variable. We have that $L_{H_1} \le L$ on $(\mathbb{R}^d)^{N-1} \times \Omega$. It follows that

$$\begin{aligned}
\sum_{i=1}^{N-1} \langle u_i(x), S^i(x)\rangle &\le L_{H_1}(x, u_1(x), ..., u_{N-1}(x)) + H_1(x, Sx, ..., S^{N-1}x) \\
&\le L(x, u_1(x), ..., u_{N-1}(x)) + H_1(x, Sx, ..., S^{N-1}x) \\
&= \sum_{i=1}^{N-1} \langle u_i(x), S^i(x)\rangle,
\end{aligned}$$

from which we deduce that

$$\sum_{i=1}^{N-1} \langle u_i(x), S^i(x)\rangle = L_{H_1}(x, u_1(x), ..., u_{N-1}(x)) + H_1(x, Sx, ..., S^{N-1}x),$$

and

$$\int_\Omega \sum_{i=1}^{N-1} \langle u_i(x), S^i(x)\rangle\, dx = \int_\Omega L_{H_1}(x, u_1(x), ..., u_{N-1}(x))\, dx + \int_\Omega H_1(x, Sx, ..., S^{N-1}x)\, dx.$$

Use now the optimality of H_1 compared to $H_1 - rH_1$ for $-1 < r < 1$ (Indeed, the above equality will be an inequality when H_1 is replaced by $H_1 - rH_1$ for $r \neq 0$) and the same argument as in the proof of the existence part in Theorem 1.1 for H_∞ to obtain that $\int_\Omega H_1(x, Sx, ..., S^{N-1}x)\, dx = 0$. On the other hand, we have

$$\int_\Omega \sum_{i=1}^{N-1} \langle u_i(x), S^i(x)\rangle\, dx \le MK' = DK \le \int_\Omega L_{H_1}(x, u_1(x), ..., u_{N-1}(x))\, dx,$$

which yields

$$\int_\Omega \sum_{i=1}^{N-1} \langle u_i(x), S^i(x)\rangle\, dx = MK' = DK = \int_\Omega L_{H_1}(x, u_1(x), ..., u_{N-1}(x))\, dx.$$

Now we can show that $u_i(x) \in \partial_{i+1} H_\infty(x, Sx, ..., S^{N-1}x)$ a.e. In fact,

$$\begin{aligned}
\int_\Omega \sum_{i=1}^{N-1} \langle u_i(x), S^i(x)\rangle\, dx &= \int_\Omega L_{H_1}(x, u_1(x), ..., u_{N-1}(x))\, dx \\
&= DK = \int_\Omega L_\infty(x, u_1(x), ..., u_{N-1}(x))\, dx \\
&\ge \int_\Omega L_\infty(x, u_1(x), ..., u_{N-1}(x))\, dx + \int_\Omega H_\infty(x, Sx, ..., S^{N-1}x)\, dx \\
&\ge \int_\Omega \sum_{i=1}^{N-1} \langle u_i(x), S^i(x)\rangle\, dx,
\end{aligned}$$

which implies that

$$\textstyle\sum_{i=1}^{N-1} \langle u_i(x), S^i(x)\rangle = L_\infty(x, u_1(x), ..., u_{N-1}(x)) + H_\infty(x, Sx, ..., S^{N-1}x) \text{ a.e. on } \Omega,$$

and hence the desired result. □

Lemma 5.2 *Suppose S is a measure preserving N-involution and $u_i(x) = \nabla_{i+1} H_\infty(x, Sx, ..., S^{N-1})$ a.e. for $i = 1, ..., N-1$. Then*

$$\nabla_1 H_\infty(x, Sx, ..., S^{N-1}x) = -\sum_{i=1}^{N-1} u_i(S^{N-i}x) \qquad a.e.\ x \in \Omega$$

Proof. Let $u \in \mathbb{R}^d$ and let $|t|$ be small. Note that

$$\int_\Omega \sum_{i=1}^{N} H_\infty\big(\sigma^{N+1-i}(x, Sx, ..., S^{N-1}x)\big)\, dx = N \int_\Omega H_\infty\big(x, Sx, ..., S^{N-1}x\big)\, dx = 0.$$

Since $\sum_{i=1}^N H_\infty(\sigma^{N+1-i}\big(x, Sx, ..., S^{N-1}(x)\big)) \le 0$, it follows that

$$\sum_{i=1}^{N} H_\infty\big(\sigma^{N+1-i}(x, Sx, ..., S^{N-1}x)\big) = 0 \quad \text{ a.e. } x \in \Omega.$$

Note that H_∞ is N-sub-antisymmetric and therefore

$$\sum_{i=1}^{N} H_\infty\big(\sigma^{N+1-i}(x + tu, Sx, ..., S^{N-1}x)\big) \le 0 = \sum_{i=1}^{N} H_\infty\big(\sigma^{N+1-i}(x, Sx, ..., S^{N-1}x)\big).$$

Assuming x is a point where $\nabla_i H_\infty\big(\sigma^{N+1-i}(x + tu, Sx, ..., S^{N-1}x)\big)$ exists for all $i = 1, ..., N$, then

$$\sum_{i=1}^{N} \nabla_i H_\infty\big(\sigma^{N+1-i}(x, Sx, ..., S^{N-1}x)\big) = 0.$$

Since $u_i(x) = \nabla_{i+1} H_\infty(x, Sx, .., S^{N-1}x)$ and $S^N = I$ a.e., we have for $i = 2, 3, ..., N$,

$$u_{i-1}(S^{N+1-i}x) = \nabla_i H_\infty\big(\sigma^{N+1-i}(x, Sx, ..., S^{N-1}x)\big).$$

Therefore,

$$\sum_{i=1}^{N-1} u_i(S^{N-i}x) + \nabla_1 H_\infty(x, Sx, ..., S^{N-1}x) = 0.$$

Proposition 5.1 *Let $u_1, ..., u_{N-1}$ be vector fields in $W^{1,1}_{loc}(\Omega)$ such that for any two families of points $x_1, ..., x_N$ and $y_1, ..., y_N$ in Ω, the function*

$$x \to \sum_{i=1}^{N-1} \langle u_i(x), y_i - x_i\rangle + \sum_{i=1}^{N-1} \langle u_i(y_{N-i}) - u_i(x_{N-i}), x\rangle$$

has no critical point unless when $x_1 = y_1$. Then, there is a unique measure preserving N-involution S on Ω that satisfies (15) *for some concave-convex N-sub-antisymmetric Hamiltonian H.*

Proof. Suppose S_1, S_2 are two measure preserving N-involutions on Ω and H_1 and H_2 are two concave-convex N-sub-antisymmetric Hamiltonian on Ω^N such that for $j = 1, 2$, we have

$$u_i(x) = \nabla_i H_j(x, S^1_j x, ..., S^{N-1}_j) \quad i = 1, ..., N-1. \tag{43}$$

We shall show that $S_1 = S_2$ a.e. on Ω. Note first that Lemma 5.1 gives that

$$u_i(x) = \nabla_i H_\infty(x, S^1_j x, ..., S^{N-1}_j). \tag{44}$$

From the preceding lemma, we have that

$$-\sum_{i=1}^{N-1} u_i(S^{N-i}_j x) = \nabla_1 H_\infty(x, S^1_j x, ..., S^{N-1}_j).$$

Note that the function $x \to L_\infty(x, u_1, ..., u_{N-1}(x))$ is locally Lipschitz and therefore is differentiable on a subset Ω_0 of full measure. We now show that $S_1 = S_2$ on Ω_0.

Indeed, for any $x \in \Omega_0$, $h = 0$ is a minimum for the function

$$h \to L_\infty(x+h, u_1(x+h), ..., u_{N-1}(x+h)) + H_\infty(x+h, S^1_j x, ..., S^{N-1}_j x) - \sum_{i=1}^{N-1} \langle u_i(x+h), S^i_j(x)\rangle.$$

This implies that

$$\begin{aligned}\nabla_1 H_\infty(x, S^1_1 x, ..., S^{N-1}_1 x) - \sum_{i=1}^{N-1} \langle \nabla u_i(x), S^i_1(x)\rangle &= -\frac{d}{dh} L_\infty(x+h, u_1(x+h), ..., u_{N-1}(x+h))_{h=0} \\ &= \nabla_1 H_\infty(x, S^1_2 x, ..., S^{N-1}_2 x) - \sum_{i=1}^{N-1} \langle \nabla u_i(x), S^i_2(x)\rangle.\end{aligned}$$

This yields that

$$\begin{aligned}\sum_{i=1}^{N-1} \langle \nabla u_i(x), S^i_2(x) - S^i_1(x)\rangle &= \nabla_1 H_\infty(x, S^1_2 x, ..., S^{N-1}_2 x) - \nabla_1 H_\infty(x, S^1_1 x, ..., S^{N-1}_1 x) \\ &= \sum_{i=1}^{N-1} \Big(u_i(S^{N-i}_1(x)) - u_i(S^{N-i}_2(x)\Big).\end{aligned}$$

The hypothesis then implies that $S_1(x) = S_2(x)$, and S is therefore unique.

In order to find examples of families of vector fields satisfying the above sufficient condition for uniqueness, we look again at N-monotone vector fields. For that we introduce the following notion.

Definition 5.3 *Say that a family of vector fields $(u_1, u_2, ..., u_{N-1})$ on Ω is strictly jointly N-monotone if*

$$\sum_{i=1}^{N}\sum_{l=1}^{N-1}\langle u_l(x_i), x_i - x_{i+l}\rangle > 0, \tag{45}$$

for every cycle $x_1, ..., x_{2N-1}$ of points in Ω such that $x_{N+l} = x_l$ for $1 \le l \le N-1$, and $x_1 \neq x_2$.

Note that for $N = 2$, this property means that the vector field u_1 is *strictly* 2*-monotone*, meaning that

$$\langle u_1(y) - u_1(x), y - x\rangle > 0 \text{ for all } y, x \in \Omega \text{ with } x \neq y. \tag{46}$$

In this case, it is easy to see that if u_1 is differentiable, then strict monotonicity implies the sufficient condition for uniqueness mentioned in Proposition 5.1. Indeed, let $u \in \mathbb{R}^d$ and $x \in \Omega$. By taking $y = x + tu$ in (46) and letting $t \to 0^+$ we obtain $\langle \nabla u_1(x)u, u\rangle \ge 0$.

Assume now that the function $x \to \langle u_1(x), y_1 - x_1\rangle + \langle u_1(y_1) - u_1(x_1), x\rangle$ has a critical point and that $y_1 \neq x_1$. It follows that

$$\langle \nabla u_1(x)(y_1 - x_1), y_1 - x_1\rangle + \langle u_1(y_1) - u_1(x_1), y_1 - x_1\rangle = 0.$$

Since the first term is non-negative and the second one is strictly positive, this leads to a contradiction.

One can however, establish directly the following uniqueness result for strictly jointly N-monotone families for $N \ge 2$, even without the differentiability assumption on $u_1, ..., u_{N-1}$. This is because we already know from the result of Galichon-Ghoussoub mentioned above that $S_1(x) = x$ is one of the possible N-involution measure preserving maps in the representation of $(u_1, ..., u_{N-1})$.

Proposition 5.2 *Assume $u_1, ..., u_{N-1}$ is a strictly jointly N-monotone family of bounded vector fields on Ω. Then, $S = I$ is the only measure preserving N-involution S on Ω that satisfies* (15) *for some concave-convex N-sub-antisymmetric Hamiltonian H.*

Proof. Let's assume S is another measure preserving N-involution in the decomposition. Let $x_i = S^i x$ for $i = 1, 2, ..., N$ and note that $x_N = x$. It follows from (45) that

$$\sum_{i=0}^{N-1}\sum_{l=1}^{N-1}\langle u_l(S^i x), S^i x - S^{i+l}x\rangle \ge 0.$$

Integrating the above expression over Ω implies that

$$\begin{aligned}
0 &\le \int_\Omega \sum_{i=0}^{N-1}\sum_{l=1}^{N-1}\langle u_l(S^i x), S^i x - S^{i+l}x\rangle\, dx \\
&= \sum_{i=0}^{N-1}\sum_{l=1}^{N-1}\int_\Omega \langle u_l(S^i x), S^i x\rangle\, dx - \sum_{i=0}^{N-1}\sum_{l=1}^{N-1}\int_\Omega \langle u_l(S^i x), S^{i+l}x\rangle\, dx \\
&= \sum_{i=0}^{N-1}\sum_{l=1}^{N-1}\int_\Omega \langle u_l(x), x\rangle\, dx - \sum_{i=0}^{N-1}\sum_{l=1}^{N-1}\int_\Omega \langle u_l(x), S^l x\rangle\, dx \\
&= N\sum_{l=1}^{N-1}\int_\Omega \langle u_l(x), x\rangle\, dx - N\sum_{l=1}^{N-1}\int_\Omega \langle u_l(x), S^l x\rangle\, dx \\
&= N\int_\Omega L_\infty(x, u_1(x), ..., u_{N-1}(x))\, dx - N\int_\Omega L_\infty(x, u_1(x), ..., u_{N-1}(x))\, dx \\
&= 0.
\end{aligned}$$

The latter is because both terms correspond to the optimal value (MK'). Since the integrand in the first line of the above expression is nonnegative we obtain

$$\sum_{i=0}^{N-1}\sum_{l=1}^{N-1}\langle u_l(S^i x), S^i x - S^{i+l}x\rangle = 0, \qquad \text{a.e. } x \in \Omega,$$

and therefore $Sx = x$. $\square$

6 Proof of Theorem 1.2

The question here is what happens when some of the vector fields u_i are identically zero. Let us illustrate the situation by assuming that just one of them, say $u_{N-1} \equiv 0$. In this case, there are two scenarios:

(I) One can begin with $N-2$ vectors $u_1, .., u_{N-2}$, and obtain a sub $(N-1)-$antisymmetric hamiltonian H and an $(N-1)-$involution S such that $u_i(x) \in \partial H_{i+1}(x, Sx, ..., S^{N-2}x)$.

(II) One can proceed as above, while considering $u_{N-1} \equiv 0$ a vector field like the others. Note that in the proof of the main theorem we never assumed $u_{N-1} \neq 0$, except on line (42) and the preceding paragraph. However, it is easily seen that by assuming $u_{N-1} \equiv 0$, one still gets

$$(u_1(x), ..., u_{N-2}(x)) \in \partial_{2,...,N-2} H_\infty(x, Sx, ..., S^{N-1}x) \qquad a.e. \quad x \in \Omega.$$

and the dependence of the Hamiltonian H_∞ with respect to the N-th variable seems to be redundant. It is interesting to see that in this case H_∞ can be chosen to be an $N-$antisymmetric Hamiltonian, which depends on only $N-1$ variables.

Indeed, we shall show that $H^1_{reg}(x_1, ..., x_N) = H_{L_H^{**}}(x_1, , ..., x_{N-1}, x_N)$ can be replaced by

$$H^0_{reg}(x_1, ..., x_{N-1}, x_N) := F_0(x_1, ..., x_{N-1}), \tag{47}$$

where

$$F_0(x_1, , ..., x_{N-1}) = \sup_{p_2,...,p_{N-1} \in B_R} \Big\{ \sum_{i=2}^{N-1} \langle p_i, x_i \rangle - L_H^{**}(x_1, p_2, ..., p_{N-1}, 0) \Big\}.$$

It follows from (31) that for all $x_i \in \Omega$ and $p^i_j \in \mathbb{R}^d$ the following inequality holds

$$\sum_{i=1}^N \Big\{ \sum_{j=1, j\neq i}^N \langle p^i_j, x_j \rangle - L_H^{**}(R^{i-1}(p^i_1, ..., p^i_{i-1}, x_i, p^i_{i+1}, ..., p^i_N)) \Big\} \leq 0.$$

In the above expression, set $p^i_{i-1} = p^1_N = 0$ for $i > 1$. By taking sup over all non-zero $p^j_i \in B_R$ we obtain

$$\sum_{i=1}^N H^0_{reg}\big(\sigma^{i-1}(x_1, ..., x_N)\big) \leq 0. \tag{48}$$

This proves that H^0_{reg} is $N-$sub-antisymmetric. By defining

$$H^2_{reg}(\mathbf{x}) = \frac{(N-1)H^0_{reg}(\mathbf{x}) - \sum_{i=1}^{N-1} H^0_{reg}(\sigma^i(\mathbf{x}))}{N},$$

and using a similar argument as in the proof of Proposition 3.1, one can also obtain that $L_{H^2_{reg}} \leq L_{H^0_{reg}} \leq L_H$ on $\bar{\Omega} \times (B_R)^{N-1}$. This together with (47) and (48) imply that the Hamiltonian H_∞ obtained in the proof of Theorem 1.1 can be chosen to be independent with respect to the last variable.

Similarly, one can show that if more than one vector fields is zero, then the dependence of H_∞ on the corresponding variables can be dropped.

Suppose now that $u_2 = ... = u_{N-1} = 0$. In this case H_∞ is just function of two variables, i.e. $H(x_1, x_2, ..., x_N) = F(x_1, x_2)$ for some Lipschitz function F, which is concave with respect to the first variable and convex with respect to the second one. Therefore $u_1(x) \in \partial_2 F(x, Sx)$ for some measure preserving $N-$involution. In this case, H_∞ being sub N-antisymmetric reads as

$$\sum_{i=1}^N F(x_{i+1}, x_i) \leq 0 \text{ for all } x_1, ...x_N \in \Omega \text{ with } x_1 = x_{N+1}.$$

7 Appendix

Theorem 7.1 *Consider vector fields $(u_i)_{i=1}^{N-1}$ on Ω such that for $i = 1, 2, ..., N-1$,*

$$u_i(x) \in \partial_{i+1} H(x, Sx, ..., S^{N-1}x) \ a.e. \ \Omega, \tag{49}$$

where $S : \bar{\Omega} \to \bar{\Omega}$ is a measure preserving N-involution, and $H : \mathbb{R}^d \times (\mathbb{R}^d)^{N-1}$ is a Lipschitz function satisfying the following properties:

1. *$H(\,.\,, X)$ is concave for every $X \in (\mathbb{R}^d)^{N-1}$, and $H(x, .\,)$ is convex for all $x \in \mathbb{R}^d$.*
2. *H is N-sub-antisymmetric on $(\bar{\Omega})^N$.*
3. *$\int_\Omega H(x, Sx, ..., S^{N-1}x)\, dx = 0$.*

If either $S \in W^{1,1}_{loc}(\Omega))$ or $u_i \in W^{1,1}_{loc}(\Omega)$ for $i = 1, 2, ..., N-1$, then there exists a full measure subset Ω_0 of Ω such that $\nabla_i H(x, Sx, ..., S^{N-1}x)$ exists for all $x \in \Omega_0$.

We shall need a few preliminary results. We first list some of the properties of directional derivatives of convex functions.

Lemma 7.2 *Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a proper convex function. Let x be a point where f is finite. The following statements hold:*

1. *For each $v \in \mathbb{R}^n$, the difference quotient in the definition of $Df(x)v$ is a non-decreasing function of $\lambda > 0$, so that $Df(x)v$ exists and*
$$Df(x)v = \inf_{\lambda>0} \frac{f(x + \lambda v) - f(x)}{\lambda}. \tag{50}$$
2. *the function $v \to Df(x)v$ is a positively homogeneous convex function of v with*
$$Df(x)u + Df(x)(-v) \geq 0 \qquad \forall v \in \mathbb{R}^n.$$

Lemma 7.3 *For each $v \in \mathbb{R}^d$, we have*

$$\int_\Omega D_1 H(x, Sx, S^2x, ..., S^{N-1}x)(v)\, dx + \int_Q D_1 H(x, Sx, S^2x, ..., S^{N-1}x)(-v)\, dx = 0.$$

Proof. Let $t > 0$ and define

$$\begin{aligned} I^1(x, v, t) &= H(x, S(x + tv), S^2(x + tv), ..., S^{N-1}(x + tv)), \\ I^2(x, v, t) &= H(x + tv, Sx, S^2x, ..., S^{N-1}x). \end{aligned}$$

Let $g \in C_c^\infty(\Omega)$ be a non-negative function. By a simple change of variables, we have for $t > 0$ small enough,

$$\int_\Omega \frac{I^1(x, v, t)g(x) + I^1(x, -v, t)g(x) - 2I^1(x, 0, 0)g(x)}{t}\, dx = \int_\Omega \frac{I^2(x, -v, t)g(x - tv) + I^2(x, v, t)g(x + tv) - 2I^1(x, 0, 0)g(x)}{t}\, dx \tag{51}$$

The limit of the right hand side of the above expression exists as $t \to 0^+$ and

$$\lim_{t\to 0^+} \int_\Omega \frac{I^2(x, -v, t)g(x - tv) + I^2(x, v, t)g(x + tv) - 2I^1(x, 0, 0)g(x)}{t}\, dx = \int_\Omega \Big[D_1 H(x, Sx, S^2x, ..., S^{N-1}x)(v) + D_1 H(x, Sx, S^2x, ..., S^{N-1}x)(-v)\Big] g(x)\, dx \leq 0, \tag{52}$$

where the last inequality is due to the concaveness of H with respect to the first variable. We shall now prove that the limit of the left hand side of (51) is non-negative as $t \to 0^+$. It follows from the convexity of H with respect to the last $N-1$ variable together with $u_i(x) \in \partial_{i+1} H(x, Sx, ..., S^{N-1}x)$ that

$$\int_\Omega \frac{I^1(x,v,t)g(x) + I^1(x,-v,t)g(x) - 2I^1(x,0,0)g(x)}{t}\,dx \geq$$

$$\frac{1}{t}\int_\Omega \sum_{i=1}^{N-1}\langle u_i(x), S^i(x+tv) + S^i(x-tv) - 2S(x)\rangle g(x)\,dx$$

The right hand side of the above expression goes to zero, as $t \to 0$, provided either $S \in W^{1,1}_{loc}(\Omega)$ or $u_i \in W^{1,1}_{loc}(\Omega)$ for $i = 1, 2, ..., N-1$. This together with (51) and (52) imply that

$$\int_\Omega \Big[D_1H(x, Sx, S^2x, ..., S^{N-1}x)(v) + D_1H(x, Sx, S^2x, ..., S^{N-1}x)(-v)\Big]g(x)\,dx = 0,$$

from which the desired results follows. □

Lemma 7.4 *For $v \in \mathbb{R}^d$, define $G_i(v) = \int_\Omega D_iH(x, Sx, ..., S^{N-1}x)(v)\,dx$. Then*

$$\sum_{i=1}^{N}[G_i(v) + G_i(-v)] \leq 0.$$

Proof. Define $f_i(t,x,v) = H(\sigma^{N+1-i}\big(x+tv, Sx, ..., S^{N-1}x\big))$. Note that

$$t \to \frac{f_i(t,x,v) + f_i(t,x,-v) - 2f_i(0,x,v)}{t}$$

is monotone and does not change sign. It follows from the monotone convergence theorem that

$$\begin{aligned}
\lim_{t\to 0^+}\int_\Omega \frac{f_i(t,x,v) + f_i(t,x,-v) - 2f_i(0,x,v)}{t}\,dx &= \int_\Omega \lim_{t\to 0^+}\frac{f_i(t,x,v) + f_i(t,x,-v) - 2f_i(0,x,v)}{t}\,dx \\
&= \int_\Omega \big[D_iH(\sigma^{N+1-i}\big(x, Sx, ..., S^{N-1}x\big))(v) + D_iH(\sigma^{N+1-i}\big(x, Sx, ..., S^{N-1}x\big))(v-)\big]\,dx \\
&= \int_\Omega \big[D_iH(x, Sx, ..., S^{N-1}x)(v) + D_iH(x, Sx, ..., S^{N-1}x)(-v)\big]\,dx = G_i(v) + G_i(-v)
\end{aligned}$$

Let $\chi_\Omega(t,x)$ be a function that is one when both $x+tv, x-tv \in \Omega$ and zero otherwise. It follows from the dominated convergence theorem that

$$\begin{aligned}
G_i(v) + G_i(-v) &= \int_\Omega \lim_{t\to 0^+}\frac{f_i(t,x,v) + f_i(t,x,-v) - 2f_i(0,x,v)}{t}\chi_\Omega(t,x)\,dx \\
&= \lim_{t\to 0^+}\int_\Omega \frac{f_i(t,x,v) + f_i(t,x,-v) - 2f_i(0,x,v)}{t}\chi_\Omega(t,x)\,dx.
\end{aligned}$$

Let $f(t,x,v) = \sum_{i=1}^N f_i(t,x,v)$. Note that for each $x \in \Omega$ one has $f(t,x,v) = \sum_{i=1}^N f_i(t,x,v) \leq 0$ for t small enough such that $x + tv \in \Omega$. Similarly $f(t,x,-v) \leq 0$ for $x - tv \in \Omega$. One also has that $\int_\Omega f(0,x,v)\,dx = 0$. It follows that

$$\begin{aligned}
\sum_{i=1}^{N}[G_i(v) + G_i(-v)] &= \int_\Omega \lim_{t\to 0^+}\frac{f(t,x,v) + f(t,x,-v) - 2f(0,x,v)}{t}\chi_\Omega(t,x)\,dx \\
&= \lim_{t\to 0^+}\int_\Omega \frac{f(t,x,v) + f(t,x,-v) - 2f(0,x,v)}{t}\chi_\Omega(t,x)\,dx \\
&= \lim_{t\to 0^+}\int_\Omega \frac{f(t,x,v) + f(t,x,v)}{t}\chi_\Omega(t,x)\,dx \leq 0.
\end{aligned}$$

Proof of Theorem 7.1. It follows from Lemma 7.3 and 7.4 that for each $v \in \mathbb{R}^d$, and $i = 1, 2, ..., N$

$$\int_\Omega \Big[D_iH(x, Sx, S^2x, ..., S^{N-1}x)(v) + D_iH(x, Sx, S^2x, ..., S^{N-1}x)(-v)\Big]\,dx = 0. \tag{53}$$

Since the integrand does not change sign, it has to be zero almost everywhere. Now choose $\{v_k\}_{k=1}^{\infty}$ to be a countable dense subset of $\mathbb{R}^d$. Set

$$A_k = \{x \in \Omega; D_iH(x, Sx, S^2x, ..., S^{N-1}x)(v_k) + D_iH(x, Sx, S^2x, ..., S^{N-1}x)(-v_k) = 0, \quad 1 \le i \le N\}$$

It follows from (53) that $\Omega \setminus A_k$ is a null set. Let $\Omega_0 = \cap_k A_k$. It follows that Ω_0 is a full measure subset of Ω such that $\nabla_i H(x, Sx, S^2x, ..., S^{N-1}x)$ exists for all $x \in \Omega_0$. □

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