Selfdual variational principles for periodic solutions of Hamiltonian and other dynamical systems

Nassif Ghoussoub* and Abbas Moameni†

Department of Mathematics, University of British Columbia,

Vancouver BC Canada V6T 1Z2

nassif@math.ubc.ca

moameni@math.ubc.ca

September 20, 2005

Abstract

Selfdual variational principles are introduced in order to construct solutions for Hamiltonian and other dynamical systems which satisfy a variety of linear and nonlinear boundary conditions including many of the standard ones. These principles lead to new variational proofs of the existence of parabolic flows with prescribed initial conditions, as well as periodic, anti-periodic and skew-periodic orbits of Hamiltonian systems. They are based on the theory of anti-selfdual Lagrangians introduced and developed recently in [3], [4] and [5].

1 Introduction

The existence of a selfdual variational principle for gradient flows of convex functionals was conjectured in [1] and established in [7]. Similar selfdual variational principles were later introduced in [8] and [6] for the resolution of certain gradient and Hamiltonian flows that connect two prescribed Lagrangian submanifolds. In this paper, we introduce new anti-selfdual Lagrangians in order to construct variationally solutions of evolution equations that satisfy certain nonlinear boundary conditions. These include the more traditional ones, such as the existence of flows with prescribed initial conditions, as well as periodic, anti-periodic and skew-periodic orbits. Our first variational principle typically deals with gradient flows of the form:

$$-\dot{x}(t) = \partial\varphi\big(t, x(t)\big) \tag{1}$$

where $\varphi(t, \ )$ is a convex lower semi-continuous function on a Hilbert space H. Our second principle deals with Hamiltonian systems of the form:

$$-J\dot{x}(t) \in \partial\varphi(t, x(t)) \tag{2}$$

where here $\varphi(t,\cdot)$ is a convex lower semi-continuous functional on $H \times H$, and J is the symplectic operator defined as $J(p,q) = (-q,p)$. In both cases, the prescribed conditions can be quite general but they include as particular cases the following more traditional ones:

- an initial value problem: $x(0) = x_0$.
- a periodic orbit: $x(0) = x(T)$,
- an anti-periodic orbit: $x(0) = -x(T)$ or
- a skew-periodic orbit (in the case of a Hamiltonian system): $x(0) = Jx(T)$.

We are looking here for selfdual variational principles, and these depend closely on the scalar product of the underlying path space. The novelty here is in the introduction of appropriate boundary Lagrangians G which, together with the main Lagrangian $L(t,x,p)$, yields an anti-selfdual Lagrangian on a path space

*Partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

†Research supported by a postdoctoral fellowship at the University of British Columbia.

equipped with an adequately defined scalar product. The following space (scalar product) seems to be well adapted to our framework.

Let $[0,T]$ be a fixed real interval, and let L^2_H be the classical space of Bochner integrable functions from $[0,T]$ to H. We consider the Hilbert space $A^2_H := \{u : [0,T] \to H; \dot{u} \in L^2_H\}$ consisting of all absolutely continuous arcs $u : [0,T] \to H$ equipped with the norm

$$\|u\|_{A^2_H} = \Big\{ \big\| \frac{u(0)+u(T)}{2} \big\|^2_H + \int_0^T \|\dot{u}\|^2_H \, dt \Big\}^{\frac{1}{2}}$$

We now recall the concept of anti-selfduality introduced in [3].

Definition 1 Given a reflexive Banach space X, we say that a convex lower semi-continuous function $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ is an *anti-selfdual Lagrangian* if

$$L^*(p,x) = L(-x,-p) \quad \text{for all } (x,p) \in X \times X^*,$$

where here L^* is the Legendre transform in both variables.

A *time dependent anti-selfdual Lagrangian* on $[0,T] \times X \times X^*$ is any function $L : [0,T] \times X \times X^* \to \mathbb{R} \cup \{+\infty\}$ that is measurable with respect to the σ-field generated by the products of Lebesgue sets in $[0,T]$ and Borel sets in $X \times X^*$ and such that $L(t,\cdot,\cdot)$ is an anti-selfdual Lagrangian for every $t \in [0,T]$.

The *Hamiltonian* H_L of L is the function defined on $[0,T] \times H \times H$ by:

$$H_L(t,x,y) = \sup\{\langle y,p\rangle - L(t,x,p); p \in H\}$$

Here is our first variational principle

Theorem 1.1 *Consider a time dependent anti-selfdual Lagrangian $L(t,x,p)$ on $[0,T] \times H \times H$ where H is a Hilbert space, and let G be an anti-selfdual Lagrangian on $H \times H$. Consider on A^2_H the following functional*

$$I(x) = \int_0^T L\big(t, x(t), \dot{x}(t)\big)\, dt + G\big(x(0) - x(T), \frac{x(0)+x(T)}{2}\big).$$

Assume the following conditions hold:

(A_1) $\quad -\infty < \int_0^T L(t,x(t),0)\, dt \le C\big(1 + \|x\|^2_{L^2_H}\big), \quad x \in L^2_H.$

(A_2) $\quad \int_0^T H_L(t,0,x(t))\, dt \to +\infty$ *as* $\|x\|_{L^2_H} \to +\infty.$

(A_3) $\quad$ *G is bounded from below and* $0 \in \mathrm{Dom}_1(G)$.

Then, there exists $\hat{x} \in A^2_H$ such that

$$I(\hat{x}) = \inf_{x \in A^2_H} I(x) = 0 \tag{3}$$

$$\big(-\dot{\hat{x}}(t), -\hat{x}(t)\big) \in \partial L\big(t, \hat{x}(t), \dot{\hat{x}}(t)\big) \quad \text{for all } t \in [0,T] \tag{4}$$

$$\big(-\frac{\hat{x}(0)+\hat{x}(T)}{2}, \hat{x}(T) - \hat{x}(0)\big) \in \partial G\big(\hat{x}(0) - \hat{x}(T), \frac{\hat{x}(0)+\hat{x}(T)}{2}\big). \tag{5}$$

The most basic time-dependent anti-selfdual Lagrangians are of the form $L(t,x,p) = \varphi(t,x) + \varphi^*(t,-p)$ where for each t, the function $x \to \varphi(t,x)$ is convex and lower semi-continuous on X. Let now $\psi : H \to \mathbb{R} \cup \{+\infty\}$ be another convex lower semi-continuous function. The above principle then yields that if $-C \le \int_0^T \varphi\big(t,x(t)\big)\, dt \le C\big(\|x\|^2_{L^H_2} + 1\big)$ and $\Phi(t,x) := \varphi(t,x) + \frac{w}{2}|x|^2_H + \langle f(t), x\rangle$, then the infimum of the functional

$$I(x) = \int_0^T \Phi(t,x(t)) + \Phi^*(t,-\dot{x}(t))\, dt + \psi(x(0)-x(T)) + \psi^*(-\frac{x(0)+x(T)}{2})$$

on A^2_H is zero and is attained at a solution $x(t)$ of the following equation

$$\begin{aligned} -\dot{x}(t) &= \partial\varphi\big(t,x(t)\big) + wx(t) + f(t) \quad \text{for all } t \in [0,T] \\ -\frac{x(0)+x(T)}{2} &\in \partial\psi(x(0)-x(T)). \end{aligned}$$

As to the various boundary conditions, we have to choose ψ accordingly.

- Initial boundary condition $x(0) = x_0$ for a given $x_0 \in H$, then $\psi(x) = \frac{1}{4}\|x\|^2_H - \langle x, x_0\rangle$.

- Periodic solutions $x(0) = x(T)$, then ψ is chosen as:

$$\psi(x) = \begin{cases} 0 & x = 0 \\ +\infty & \text{elsewhere.} \end{cases}$$

- Anti periodic solutions $x(0) = -x(T)$, then $\psi(x) = 0$ for each $x \in H$.

It is worth noting that while the main Lagrangian L is expected to be smooth and hence its subdifferential coincides with its gradient –and the differential inclusion is often an equation, it is crucial that the boundary Lagrangian G be allowed to be degenerate so as its subdifferential can cover the boundary conditions discussed above.

For the case of Hamiltonian systems we consider for simplicity $H = \mathbb{R}^N$ and let $X = H \times H$. We shall establish the following principle.

Theorem 1.2 *Let $\varphi : [0,T] \times X \to \mathbb{R}$ be such that $(t,u) \to \varphi(t,u)$ is measurable in t for each $u \in X$, and convex and lower semi-continuous in u for a.e. $t \in [0,T]$. Let $\psi : X \to \mathbb{R} \cup \{\infty\}$ be convex and lower semi continuous on X and assume the following conditions:*

(B_1) There exists $\beta \in (0, \frac{\pi}{2T})$ and $\gamma, \alpha \in L^2(0,T;\mathbb{R}_+)$ such that $-\alpha(t) \leq \varphi(t,u) \leq \frac{\beta}{2}|u|^2 + \gamma(t)$ for every $u \in H$ and all $t \in [0,T]$.

(B_2) $\int_0^T \varphi(t,u)\,dt \to +\infty$ as $|u| \to +\infty$.

(B_3) ψ is bounded from below and $0 \in Dom(\psi)$.

(1) The infimum of the functional

$$\begin{aligned} J_1(u) &= \int_0^T [\varphi(t,u(t)) + \varphi^*(t,-J\dot{u}(t)) + \langle J\dot{u}(t), u(t)\rangle]\,dt \\ &\quad + \langle u(T) - u(0), J\frac{u(0)+u(T)}{2}\rangle + \psi\big(u(T)-u(0)\big) + \psi^*\big(-J\frac{u(0)+u(T)}{2}\big) \end{aligned}$$

on A_X^2 is then equal to zero and is attained at a solution of

$$\begin{cases} -J\dot{u}(t) &= \partial\varphi\big(t,u(t)\big), \\ -J\frac{u(T)+u(0)}{2} &\in \partial\psi\big(u(T)-u(0)\big). \end{cases} \tag{6}$$

(2) The infimum of the functional

$$J_2(u) = \int_0^T \big[\varphi\big(t,u(t)\big) + \varphi^*(t,-J\dot{u}(t)) + \langle J\dot{u}(t), u(t)\rangle\big]\,dt + \big(Ju(0), u(T)\big) + \psi\big(u(0)\big) + \psi^*\big(Ju(T)\big)$$

on A_X^2 is also zero and is attained at a solution of

$$\begin{cases} -J\dot{u}(t) &= \partial\varphi\big(t,u(t)\big), \\ Ju(T) &\in \partial\psi\big(u(0)\big). \end{cases} \tag{7}$$

In the applications, ψ is to be chosen according to the required boundary conditions. For example:

- Initial boundary condition $x(0) = x_0$ for a given $x_0 \in H$. Use the functional J_1 with $\bar{\varphi}(t,x) = \varphi(t, x - x_0)$ and $\psi(x) = 0$ at 0 and $+\infty$ elsewhere.
- Periodic solutions $x(0) = x(T)$, or more generally $x(0) - x(T) \in K$ where K is a closed convex subset of $H \times H$. Use the functional J_1 with ψ chosen as:

$$\psi(x) = \begin{cases} 0 & x \in K \\ +\infty & \text{elsewhere.} \end{cases}$$

- Anti-periodic solutions $x(0) = -x(T)$. Use the functional J_1 with $\psi(x) = 0$ for each $x \in H$.
- Skew-periodic solutions $x(0) = Jx(T)$. Use the functional J_2 with $\psi(x) = \frac{1}{2}|x|^2$.

Section 2 deals with gradient flows and the proof of Theorem 1.1, while section 3 is concerned with Hamiltonian systems. This paper is self-contained but should be read in conjunction with [3], [4] and [7] which introduce selfduality and [6] which deals with Hamiltonian systems that link Lagrangian submanifolds.

2 Gradient flows with general boundary conditions

2.1 Anti-selfdual Lagrangians on path space

We now show how a boundary anti-self dual Lagrangian allows us to "lift" a time-dependent anti-selfdual Lagrangian to the path space A^2_H. Note that we can and will identify the space A^2_H with the product space $H\times L^2_H$, in such a way that its dual $(A^2_H)^*$ can also be identified with $H\times L^2_H$ via the formula

$$\langle u,(p_1,p_0)\rangle_{A^2_H, H\times L^2_H} = \langle \frac{u(0)+u(T)}{2}, p_1\rangle + \int_0^T \langle \dot u(t), p_0(t)\rangle\, dt$$

where $u\in A^2_H$ and $(p_1,p_0(t))\in H\times L^2_H$.

Proposition 2.1 *Suppose L is an anti-self dual Lagrangian on $[0,T]\times H\times H$ and that G is an anti-selfdual Lagrangian on $H\times H$, then the Lagrangian defined on $A^2_H\times (A^2_H)^* = A^2_H\times (H\times L^2_H)$ by*

$$\mathcal{M}(u,p) = \int_0^T L\big(t, u(t)+p_0(t), \dot u(t)\big)\, dt + G\big(u(0)-u(T)+p_1, \frac{u(0)+u(T)}{2}\big)$$

is anti-self dual Lagrangian on $A^2_H\times (L^2_H\times H)$.

Proof: For $(q,v)\in A^2_H\times (A^2_H)^*$ with q represented by $(q_0(t),q_1)$ we have

$$\begin{aligned}\mathcal{M}^*(q,v) &= \sup_{p_1\in H}\sup_{p_0\in L^2_H}\sup_{u\in A^2_H}\Big\{\langle p_1, \frac{v(0)+v(T)}{2}\rangle + \langle q_1, \frac{u(0)+u(T)}{2}\rangle \\ &\quad + \int_0^T \big[\langle p_0(t), \dot v(t)\rangle + \langle q_0(t), \dot u\rangle - L\big(t, u(t)+p_0(t), \dot u(t)\big)\big]\, dt \\ &\quad - G\big(u(0)-u(T)+p_1, \frac{u(0)+u(T)}{2}\big)\Big\},\end{aligned}$$

making a substitution $u(0)-u(T)+p_1 = a\in H$ and $u(t)+p_0(t) = y(t)\in L^2_H$ we obtain

$$\begin{aligned}\mathcal{M}^*(q,v) &= \sup_{a\in H}\sup_{y\in L^2_H}\sup_{u\in A^2_H}\Big\{\langle a+u(T)-u(0), \frac{v(0)+v(T)}{2}\rangle + \langle q_1, \frac{u(0)+u(T)}{2}\rangle \\ &\quad + \int_0^T \big[\langle y(t)-u(t), \dot v\rangle + \langle q_0(t), \dot u(t)\rangle - L\big(t, y(t), \dot u(t)\big)\big]\, dt \\ &\quad - G\big(a, \frac{u(0)+u(T)}{2}\big)\Big\}.\end{aligned}$$

Since $\dot u$ and $\dot v\in L^2_H$, we have: $\int_0^T \langle u, \dot v\rangle = -\int_0^T \langle \dot u, v\rangle + \langle u(T), v(T)\rangle - \langle v(0), u(0)\rangle$ which implies

$$\begin{aligned}\mathcal{M}^*(q,v) &= \sup_{a\in H}\sup_{y\in L^2_H}\sup_{u\in A^2_H}\Big\{\langle a, \frac{v(0)+v(T)}{2}\rangle + \langle u(T), \frac{v(0)+v(T)}{2} - v(T)\rangle \\ &\quad + \langle u(0), v(0) - \frac{v(0)+v(T)}{2}\rangle + \langle q_1, \frac{u(0)+u(T)}{2}\rangle \\ &\quad + \int_0^T \big[\langle y(t), \dot v\rangle + \langle \dot u(t), v(t)+q_0(t)\rangle - L\big(t, y(t), \dot u(t)\big)\big]\, dt \\ &\quad - G\big(a, \frac{u(0)+u(T)}{2}\big)\Big\}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{M}^*(q,v) &= \sup_{a\in H}\sup_{y\in L^2_H}\sup_{u\in A^2_H}\Big\{\langle a, \frac{v(0)+v(T)}{2}\rangle + \langle q_1 + v(0) - v(T), \frac{u(0)+u(T)}{2}\rangle - G\big(a, \frac{u(0)+u(T)}{2}\big) \\ &\quad + \int_0^T \big[\langle y(t), \dot v(t)\rangle + \langle \dot u(t), v(t)+q_0(t)\rangle - L\big(t, y(t), \dot u(t)\big)\big]\, dt\Big\}.\end{aligned}$$

Identify now A^2_H with $H\times L^2_H$ via the correspondence:

$$\begin{aligned}\big(b,f(t)\big) &\in H\times L^2_H \longmapsto b+\frac{1}{2}\left(\int_t^T f(s)\,ds-\int_0^t f(s)\,ds\right)\in A^2_H,\\ u &\in A^2_H\longmapsto \big(\frac{u(0)+u(T)}{2},-\dot u(t)\big)\in H\times L^2_H.\end{aligned}$$

We finally obtain

$$\begin{aligned}\mathcal{M}^*(q,v) &= \sup_{a\in H}\sup_{b\in H}\Big\{\langle a,\frac{v(0)+v(T)}{2}\rangle+\langle q_1+v(0)-v(T),b\rangle-G(a,b)\Big\}\\ &\quad+\sup_{y\in L^2_H}\sup_{r\in L^2_H}\Big\{\int_0^T\big[\langle y(t),\dot v(t)\rangle+\langle v(t)+q_0(t),r(t)\rangle-L\big(t,y(t),r(t)\big)\big]\,dt\Big\}\\ &= G^*\big(\frac{v(0)+v(T)}{2},q_1+v(0)-v(T)\big)+\int_0^T L^*\big(t,\dot v(t),v(t)+q_0(t)\big)\,dt\\ &= G\big(-q_1-v(0)+v(T),\frac{-v(0)-v(T)}{2}\big)+\int_0^T L\big(t,-v(t)-q_0(t),-\dot v(t)\big)\,dt\\ &= \mathcal{M}(-v,-q).\end{aligned}$$

2.2 Variational principles for gradient flows with general boundary conditions

We now recall from [3] the following general result about minimizing anti-selfdual Lagrangians.

Proposition 2.2 *Let $\mathcal{M}$ be a an anti-selfdual Lagrangian on a reflexive Banach space $X\times X^*$ such that for some $x_0\in X$, the function $p\to\mathcal{M}(x_0,p)$ is bounded above on a neighborhood of the origin in X^*. Then there exists $\bar x\in X$, such that:*

$$\begin{cases}\mathcal{M}(\bar x,0) = \inf_{x\in X}\mathcal{M}(x,0)=0.\\ (0,-\bar x)\in\partial\mathcal{M}(\bar x,0).\end{cases}\tag{8}$$

We can already deduce the following version of Theorem 1.1 modulo a stronger hypothesis on the boundary Lagrangian.

Proposition 2.3 *Consider a time dependent anti-selfdual Lagrangian $L(t,x,p)$ on $[0,T]\times H\times H$ and an anti-selfdual lagrangian G on $H\times H$. Assume the following conditions:*

- *(A_1)* $\quad -\infty<\int_0^T L(t,x(t),0)\,dt\le C\big(1+\|x\|^2_{L^2_H}\big)$ *for all* $x\in L^2_H$.
- *(A_2)* $\quad G$ *is bounded from below and* $G(a,0)\le C\big(\|a\|^2_H+1\big)$ *for all* $a\in H$.

Then the functional $I(x)=\int_0^T L\big(t,x(t),\dot x(t)\big)\,dt+G\big(x(0)-x(T),\frac{x(0)+x(T)}{2}\big)$ attains its minimum at a path $\hat x\in A^2_H$ satisfying

$$I(\hat x) = \inf_{x\in A^2_H} I(x)=0\tag{9}$$

$$\big(-\dot{\hat x}(t),-\hat x(t)\big)\in\partial L\big(t,\hat x(t),\dot{\hat x}(t)\big)\quad\forall t\in[0,T]\tag{10}$$

$$\big(-\frac{\hat x(0)+\hat x(T)}{2},\hat x(T)-\hat x(0)\big)\in\partial G\big(\hat x(0)-\hat x(T),\frac{\hat x(0)+\hat x(T)}{2}\big).\tag{11}$$

Proof: Apply Proposition 2.2 to the Lagrangian

$$\mathcal{M}(u,p)=\int_0^T L\big(t,u(t)+p_0(t),\dot u(t)\big)\,dt+G\big(u(0)-u(T)+p_1,\frac{u(0)+u(T)}{2}\big)$$

which is anti-selfdual on A^2_H in view of Proposition 2.1. Noting that $I(x)=\mathcal{M}(x,0)$, we obtain $\hat x(t)\in A^2_H$ such that

$$\int_0^T L\big(t,\hat x(t),\dot{\hat x}(t)\big)\,dt+G\left(\hat x(0)-\hat x(T),\frac{\hat x(0)+\hat x(T)}{2}\right)=0,$$

which gives

$$\begin{aligned}
0 &= \int_0^T \Big[L\big(t,\hat{x}(t),\dot{\hat{x}}(t)\big)+\langle \hat{x}(t),\dot{\hat{x}}(t)\rangle\Big]\,dt-\int_0^T \langle \hat{x}(t),\dot{\hat{x}}(t)\rangle\,dt+G\big(\hat{x}(0)-\hat{x}(T),\frac{\hat{x}(0)+\hat{x}(T)}{2}\big)\\
&= \int_0^T \Big[L\big(t,\hat{x}(t),\dot{\hat{x}}(t)\big)+\langle \hat{x}(t),\dot{\hat{x}}(t)\rangle\Big]\,dt-\frac{1}{2}|\hat{x}(T)|^2+\frac{1}{2}|\hat{x}(0)|^2+G\big(\hat{x}(0)-\hat{x}(T),\frac{\hat{x}(0)+\hat{x}(T)}{2}\big)\\
&= \int_0^T L\big(t,\hat{x}(t),\dot{\hat{x}}(t)\big)+\langle \hat{x}(t),\dot{\hat{x}}(t)\rangle\,dt+\langle \hat{x}(0)-\hat{x}(T),\frac{\hat{x}(0)+\hat{x}(T)}{2}\rangle+G\big(\hat{x}(0)-\hat{x}(T),\frac{\hat{x}(0)+\hat{x}(T)}{2}\big).
\end{aligned}$$

Since $L(t,\cdot,\cdot)$ and G are anti-selfdual Lagrangians we have $L\big(t,\hat{x}(t),\dot{\hat{x}}(t)\big)+\langle \hat{x}(t),\dot{\hat{x}}(t)\rangle\geq 0$ and

$$G\big(\hat{x}(0)-\hat{x}(T),\frac{\hat{x}(0)+\hat{x}(T)}{2}\big)+\langle \hat{x}(0)-\hat{x}(T),\frac{\hat{x}(0)+\hat{x}(T)}{2}\rangle\geq 0.$$

which means that $L\big(t,\hat{x}(t),\dot{\hat{x}}(t)\big)+\langle \hat{x}(t),\dot{\hat{x}}(t)\rangle\,)=0$ for almost all $t\in[0,T]$, and

$$G\big(\hat{x}(0)-\hat{x}(T),\frac{\hat{x}(0)+\hat{x}(T)}{2}\big)+\langle \hat{x}(0)-\hat{x}(T),\frac{\hat{x}(0)+\hat{x}(T)}{2}\rangle=0.$$

The result follows from the above identities and the limiting case in Fenchel-Legendre duality.□

In order to complete the proof of Theorem 1.1, we need to perform an inf-convolution argument on the boundary Lagrangian G. We shall use the following simple estimate

Lemma 2.1 *Let $F:Y\mapsto\mathbb{R}\cup\{\infty\}$ be a proper convex and lower semi continues functional on a Banach space Y such that $-\beta\leq F(y)\leq\frac{\alpha}{p}\|y\|_Y^p+\gamma$ with $\alpha>0,p>1,\beta\geq 0$, and $\gamma\geq 0$. Then for every $y^*\in\partial F(y)$ we have*

$$\|y^*\|_{Y^*}\leq\Big\{p\alpha^{\frac{q}{p}}(\|y\|_Y+\beta+\gamma)+1\Big\}^{p-1}.$$

We shall also make frequent use of the following lemma [3].

Lemma 2.2 *Let G be an anti-selfdual Lagrangian on $X\times X^*$ and consider for each $\lambda>0$, its λ-regularization*

$$G_\lambda(x,p):=\inf\left\{G(z,p)+\frac{\|x-z\|^2}{2\lambda}+\frac{\lambda}{2}\|p\|^2;\,z\in X\right\}.$$

Then,

1. *G_λ is also an anti-selfdual Lagrangian on $X\times X^*$ and $G_\lambda(x,0)\leq G(0,0)+\frac{\|x\|^2}{2\lambda}$.*
2. *If $(0,0)\in\mathrm{Dom}(G)$ and if $x_\lambda\rightharpoonup x$ in X and $p_\lambda\rightharpoonup p$ weakly in X^* and if $G(x_\lambda,p_\lambda)$ is bounded from above, then $G(x,p)\leq\liminf_{\lambda\to 0}G_\lambda(x_\lambda,p_\lambda)$.*

Proof of Theorem 1.1: Define for each $\lambda>0$, the Lagrangian G_λ as in Lemma 2.2, and apply Proposition 2.3 to obtain $x_\lambda\in A_H^2$ such that

$$\int_0^T L\big(t,x_\lambda(t),\dot{x}_\lambda(t)\big)\,dt + G_\lambda\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big)=0 \tag{12}$$

$$\big(-\dot{x}_\lambda(t),-x_\lambda(t)\big) \in \partial L\big(t,x_\lambda(t),\dot{x}_\lambda(t)\big) \tag{13}$$

$$\big(-\frac{x_\lambda(0)+x_\lambda(T)}{2},x_\lambda(T)-x_\lambda(0)\big) \in \partial G_\lambda\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big). \tag{14}$$

We shall show that $(x_\lambda)_\lambda$ is bounded in A_H^2.

For simplicity, we shall assume that L has the form $L(t,x,p)=\varphi(t,x)+\varphi^*(t,-p)$. For such Lagrangians, Equation (13) yields that $-\dot{x}_\lambda(t)=\partial_1 L\big(t,x_\lambda(t),0\big)$. Multiply this equation by $x_\lambda(t)$ and integrate over $[0,T]\times\Omega$ to get

$$\int_0^T\langle -\dot{x}_\lambda(t),x_\lambda(t)\rangle\,dt=\int_0^T\langle\partial_1 L(t,x_\lambda(t),0),x_\lambda(t)\rangle\,dt,$$

which gives

$$-\frac{1}{2}|x_\lambda(T)|^2+\frac{1}{2}|x_\lambda(0)|^2=\int_0^T\langle\partial_1 L(t,x_\lambda(t),0),x_\lambda(t)\rangle\,dt\geq\int_0^T H_L\big(t,0,x_\lambda(t)\big)\,dt. \tag{15}$$

Also, from (14) we have

$$G_\lambda\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big)=-\langle x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\rangle=\frac{1}{2}|x_\lambda(T)|^2-\frac{1}{2}|x_\lambda(0)|^2. \tag{16}$$

Combining (15) and (16) gives that

$$G_\lambda\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big)+\int_0^T H_L\big(t,0,x_\lambda(t)\big)\,dt\leq 0. \tag{17}$$

Since G is bounded from below so is G_λ which together with condition (A_2) imply that $\int_0^T|x_\lambda(t)|^2\,dt$ is bounded.

Now from condition (A_1) and the boundedness of x_λ in L^2_H, we can apply Lemma 2.1 to get that $-\dot{x}_\lambda(t)=\partial_1 L\big(t,x_\lambda(t),0\big)$ is bounded in L^2_H. Hence, x_λ is bounded in A^2_H, thus, up to a subsequence $x_\lambda(t)\rightharpoonup\hat{x}(t)$ in A^2_H, $x_\lambda(0)\rightharpoonup\hat{x}(0)$ and $x_\lambda(T)\rightharpoonup\hat{x}(T)$ in H.

From (17), we have $G_\lambda\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big)\leq C$, and we obtain from Lemma 2.2 that

$$G\big(\hat{x}(0)-\hat{x}(T),\frac{\hat{x}(0)+\hat{x}(T)}{2}\big)\leq\liminf_{\lambda\to 0}G_\lambda\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big). \tag{18}$$

Now, if we let $\lambda\to 0$ in (12), then by considering (18) we get

$$\int_0^T L\big(t,\hat{x}(t),\dot{\hat{x}}(t)\big)\,dt+G\big(\hat{x}(0)-\hat{x}(T),\frac{\hat{x}(0)+\hat{x}(T)}{2}\big)\leq 0. \tag{19}$$

On the other hand, for every $x\in A^2_H$ we have

$$\int_0^T L\big(t,x(t),\dot{x}(t)\big)\,dt+G\big(x(0)-x(T),\frac{x(0)+x(T)}{2}\big)\ \geq 0 \tag{20}$$

which means $I(\hat{x})=0$ and as in the proof of Proposition 2.3, $x(t)$ satisfies (3), (4), and (5). □

The boundedness condition on L may be too restrictive in applications, and one may want to replace the Hilbertian norm with a stronger Banach norm for which condition $(A1)$ is more likely to hold. For this situation, we have the following result.

Theorem 2.3 *Let $X\subset H\subset X^*$ be an evolution pair and let $\psi:[0,T]\times X\to\mathbb{R}\cup\{+\infty\}$ be convex and lower semi-continuous in $x\in X$ for a.e. $t\in[0,T]$ and measurable in t for every $x\in X$. Consider the time-dependent anti-selfdual Lagrangian, $L(t,x,p)=\psi(t,x)+\psi^*(t,-p)$ on $[0,T]\times X\times X^*$ and an anti-selfdual Lagrangian G on $H\times H$. Assume the following conditions:*

- (A_1') *For some $p\geq 2$ and $C>0$, we have $-C\big(1+\|x\|^p_{L^p_X}\big)<\int_0^T L(t,x(t),0)\,dt\leq C\big(1+\|x\|^p_{L^p_X}\big)$ for every $x\in L^p_X$.*
- (A_2') *G is bounded from below, $0\in Dom(G)$ and for every $a\in H$, $G(a,b)\to+\infty$ as $\|b\|_H\to+\infty$.*

Then there exists $\hat{x}\in L^p_X$ with $\dot{\hat{x}}\in L^q_{X^*}$ ($\frac{1}{p}+\frac{1}{q}=1$), $\hat{x}(0),\hat{x}(T)\in H$ and satisfying (3), (4), and (5).

Proof: Here again we shall combine inf-convolution with Theorem 1.1. For $\lambda>0$ consider the $\lambda-$regularization of ψ,

$$\psi_\lambda(t,x)=\inf_{y\in H}\left\{\psi(t,y)+\frac{|x-y|^2_H}{2\lambda}\right\}, \tag{21}$$

where

$$\psi(t,y)=\begin{cases}\psi(t,y) & y\in X\\ +\infty & y\in H-X.\end{cases}$$

Set $L_\lambda(t,x,p)=\psi_\lambda(t,x)+\psi^*_\lambda(t,-p)$. By Theorem 1.1, there exists $x_\lambda(t)\in A^2_H$ such that

$$\int_0^T L\big(t,x_\lambda(t),\dot{x}_\lambda(t)\big)\,dt\quad+\quad G\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big)=0 \tag{22}$$

$$\big(-\dot{x}_\lambda,-x_\lambda(t)\big)\quad\in\quad\partial L\big(t,x_\lambda(t),\dot{x}_\lambda(t)\big) \tag{23}$$

$$\big(-\frac{x_\lambda(0)+x_\lambda(T)}{2},x_\lambda(T)-x_\lambda(0)\big)\quad\in\quad\partial G\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big) \tag{24}$$

We now show that $(x_\lambda)_\lambda$ is bounded in an appropriate function space. As in the proof of Theorem 1.1, we have

$$G\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big)+\int_0^T H_{L_\lambda}\big(t,0,x_\lambda(t)\big)\,dt\le 0. \tag{25}$$

Since ψ is convex and lower semi-continuous, there exists $i_\lambda(x_\lambda)$ such that the infimum in (21) attains at $i_\lambda(x_\lambda)$, i.e.

$$\psi_\lambda(t,x_\lambda)=\psi(t,i_\lambda(x_\lambda))+\frac{\|x_\lambda-i_\lambda(x_\lambda)\|^2}{2\lambda}. \tag{26}$$

Therefore,

$$\int_0^T H_{L_\lambda}\big(t,0,x_\lambda(t)\big)\,dt=\int_0^T H_L\big(t,0,i_\lambda(x_\lambda(t))\big)dt+\frac{\|x_\lambda-i_\lambda(x_\lambda)\|^2}{2\lambda}\,dt. \tag{27}$$

Plug (27) in inequality (25) to get

$$G\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big)+\int_0^T H_L\big(t,0,i_\lambda(x_\lambda(t))\big)\,dt+\frac{\|x_\lambda-i_\lambda(x_\lambda)\|^2}{2\lambda}\,dt\le 0. \tag{28}$$

By the coercivity assumptions in (A_1') , we obtain that $(i_\lambda(x_\lambda))_\lambda$ is bounded in $L^p(0,T;X)$ and $(x_\lambda)_\lambda$ is bounded in $L^2(0,T;H)$. It follows from (23) and the structure of L that $-\dot{x}_\lambda=\partial_1 L(t,i_\lambda(x_\lambda),0)$, which together with the boundedness of $(i_\lambda(x_\lambda))_\lambda$ in $L^p(0,T;X)$, condition (A_1'), and Lemma 2.1 imply that $-(\dot{x}_\lambda)_\lambda$ is bounded in $L^q(0,T;X^*)$. Also note that $x_\lambda(0)-x_\lambda(T)=\int_0^T \dot{x_\lambda}(t)\,dt$ is therefore bounded in X^*. It follows from (A_2') that $x_\lambda(0)+x_\lambda(T)$ is therefore bounded in H and so is in X^*. Hence, up to a subsequence, we have

$$i_\lambda(x_\lambda)\rightharpoonup\hat{x}\quad\text{in } L^p(0,T;X), \tag{29}$$
$$\dot{x}_\lambda\rightharpoonup\dot{\hat{x}}\quad\text{in } L^q(0,T;X^*), \tag{30}$$
$$x_\lambda\rightharpoonup\hat{x}\quad\text{in } L^2(0,T;H), \tag{31}$$
$$x_\lambda(0)\rightharpoonup\hat{x}(0)\quad\text{in } X^*, \tag{32}$$
$$x_\lambda(T)\rightharpoonup\hat{x}(T)\quad\text{in } X^*. \tag{33}$$

On the other hand it follows from (22) and (26) that

$$G\big(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\big)+\int_0^T L\big(t,i_\lambda(x_\lambda(t)),\dot{x}_\lambda\big)+\frac{\|x_\lambda-i_\lambda(x_\lambda)\|^2}{2\lambda}+\frac{\lambda}{2}\|\dot{x}_\lambda\|_H^2\,dt=0. \tag{34}$$

By letting λ go to zero in (34), we get from (29)-(33) that

$$G\big(\hat{x}_\lambda(0)-\hat{x}_\lambda(T),\frac{\hat{x}_\lambda(0)+\hat{x}_\lambda(T)}{2}\big)+\int_0^T L\big(t,\hat{x}_\lambda(t)),\dot{\hat{x}}_\lambda\big)_H^2\,dt\le 0.$$

It follows from (A_1') and the last inequality that $\hat{x}\in L^p(0,T;X)$ and $\dot{\hat{x}}\in L^q(0,T;X^*)$. The rest of the proof is similar to the proof of Proposition 2.3.

Remark 2.4 One can actually do without the coercivity condition on G in Theorem 2.3. Indeed, by using the $\lambda-$regularization G_λ of G, we get the required coercivity condition on the second variable for G_λ and we obtain from Theorem 2.3 that there exists $x_\lambda\in L^p(0,T;X)$ with $\dot{x}_\lambda\in L^q(0,T;X^*)$ such that

$$\int_0^T L\big(t,x_\lambda(t),\dot{x}_\lambda(t)\big)\,dt+G_\lambda\left(x_\lambda(0)-x_\lambda(T),\frac{x_\lambda(0)+x_\lambda(T)}{2}\right)=0. \tag{35}$$

It follows from (A_1) and the boundedness of G_λ from below that $(x_\lambda)_\lambda$ is bounded in $L^p(0,T;X)$, and since $(\dot{x}_\lambda)_\lambda$ is bounded in $L^q(0,T;X^*)$ this also means $(x_\lambda(0))_\lambda$ and $(x_\lambda(T))_\lambda$ are bounded in H. Hence, up to a subsequence we have

$$x_\lambda\rightharpoonup\hat{x}\quad\text{in } L^p(0,T;X), \tag{36}$$
$$\dot{x}_\lambda\rightharpoonup\dot{\hat{x}}\quad\text{in } L^q(0,T;X^*), \tag{37}$$
$$x_\lambda(0)\rightharpoonup\hat{x}(0)\quad\text{in } H, \tag{38}$$
$$x_\lambda(T)\rightharpoonup\hat{x}(T)\quad\text{in } H. \tag{39}$$

The rest of the proof is similar to the proof of Theorem 1.1. □

2.3 Example

As mentioned in the introduction, a typical example is

$$\left\{\begin{array}{rcl} -\dot{x}(t) & = & \partial\varphi\big(t,x(t)\big)+wx(t)+f(t) \\ x(0) & = & x_0 \quad \text{or} \quad x(0)=x(T) \quad \text{or} \quad x(0)=-x(T), \end{array}\right. \tag{40}$$

where $-C\leq\int_0^T\varphi\big(t,x(t)\big)\,dt\leq C\big(\|x\|^2_{L^H_2}+1\big)$ and $w>0$.

For the initial-value problem $x(0)=x_0$, we pick the boundary Lagrangian to be $G(x,p))=\frac{1}{4}|x|^2_H-\langle x,x_0\rangle+|x_0-p|^2$, and so the associated functional becomes

$$I(x)=\int_0^T\Phi(t,x(t))+\Phi^*(t,-\dot{x}(t))\,dt+\frac{1}{4}|x(0)-x(T)|^2-\langle x(0)-x(T),x_0\rangle+|x_0+\frac{x(0)+x(T)}{2}|^2$$

where $\Phi(t,x):=\varphi(t,x)+\frac{w}{2}|x|^2_H+\langle f(t),x\rangle$, The infimum of I on A^2_H is zero and is attained at a solution $x(t)$ of the equation. The boundary condition is then

$$-\frac{1}{2}(x(0)+x(T))=\partial_1G\big(x(0)-x(T),-\frac{x(0)+x(T)}{2}\big)=\frac{1}{2}\big(x(0)-x(T)\big)-x_0,$$

which gives that $x(0)=x_0$.

We can of course relax the conditions on φ by using again inf-convolution as was done in [7] in the case where φ is autonomous, or as in Theorem 2.3.

3 Hamiltonian systems with general boundary conditions

For a given Hilbert space H, we consider the subspace H^1_T of A^2_H consisting of all periodic functions, equipped with the norm induced by A^2_H. We also consider the space H^1_{-T} consisting of all functions in A^2_H which are anti-periodic, i.e. $u(0)=-u(T)$. The norm of H^1_{-T} is given by $\|u\|_{H^1_{-T}}=(\int_0^T|\dot{u}|^2\,dt)^{\frac{1}{2}}$. We now establish a few useful inequalities on H^1_{-T}, which can be seen as the counterparts of Wirtinger's inequality,

$$\int_0^T|u|^2\,dt\leq\frac{T^2}{4\pi^2}\int_0^T|\dot{u}|^2\,dt \quad \text{for} \quad u\in H^1_T \text{ and } \int_0^T u(t)\,dt=0,$$

and the Sobolev inequality on H^1_T,

$$\|u\|^2_\infty\leq\frac{T}{12}\int_0^T|\dot{u}|^2\,dt \quad \text{for} \quad u\in H^1_T \text{ and } \int_0^T u(t)\,dt=0.$$

Proposition 3.1 *If $u\in H^1_{-T}$ then*

$$\int_0^T|u|^2\,dt\leq\frac{T^2}{\pi^2}\int_0^T|\dot{u}|^2\,dt, \tag{41}$$

and

$$\|u\|^2_\infty\leq\frac{T}{4}\int_0^T|\dot{u}|^2\,dt. \tag{42}$$

Proof: Since $u(0)=-u(T)$, u has the Fourier expansion of the form $u(t)=\sum_{k=-\infty}^{\infty}u_k\exp((2k-1)i\pi t/T)$. The Parseval equality implies that

$$\int_0^T|\dot{u}|^2\,dt=\sum_{k=-\infty}^{\infty}T\big((2k-1)^2\pi^2/T^2\big)|u_k|^2\geq\frac{\pi^2}{T^2}\sum_{k=-\infty}^{\infty}T|u_k|^2=\frac{\pi^2}{T^2}\int_0^T|u|^2\,dt.$$

The Cauchy-Schwarz inequality and the above imply that for $t\in[0,T]$,

$$\begin{array}{rcl} |u(t)|^2 & \leq & \left(\displaystyle\sum_{k=-\infty}^{\infty}|u_k|\right)^2 \\ & \leq & \left[\displaystyle\sum_{k=-\infty}^{\infty}\frac{T}{\pi^2(2k-1)^2}\right]\left[\displaystyle\sum_{k=-\infty}^{\infty}T\big((2k-1)^2\pi^2/T^2\big)|u_k|^2\right] \\ & = & \displaystyle\frac{T}{\pi^2}\sum_{k=-\infty}^{\infty}\frac{1}{(2k-1)^2})\int_0^T|\dot{u}|^2\,dt. \end{array}$$

and conclude by noting that $\sum_{k=-\infty}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{4}$.

Proposition 3.2 *Consider the space A_X^2 where $X = H \times H$ and let J be the symplectic operator on X defined as $J(p,q) = (-q,p)$.*

1. *If H is any Hilbert space, then for every $u \in A_X^2$*

$$\Big| \int_0^T (J\dot{u}, u)\, dt + \Big(J\frac{u(0)+u(T)}{2}, u(T) - u(0)\Big) \Big| \le \frac{T}{2} \int_0^T \big|\dot{u}(t)\big|^2\, dt.$$

2. *If H is finite dimensional, then*

$$\Big| \int_0^T (J\dot{u}, u)\, dt + \Big(J\frac{u(0)+u(T)}{2}, u(T) - u(0)\Big) \Big| \le \frac{T}{\pi} \int_0^T \big|\dot{u}(t)\big|^2\, dt.$$

Proof: For part (i), note that each $u \in A_X^2$ can be written as follows,

$$u(t) = \frac{1}{2}\Big(\int_0^t \dot{u}(s)\, ds - \int_t^T \dot{u}(s)\, ds \Big) + \frac{u(0)+u(T)}{2}.$$

where $v(t) = u(t) - \frac{u(0)+u(T)}{2} = \frac{1}{2}\Big(\int_0^t \dot{u}(s)\, ds - \int_t^T \dot{u}(s)\, ds\Big)$ clearly belongs to H^1_{-T}. Multiplying both sides by $J\dot{u}$ and integrating over $[0,T]$, we get

$$\int_0^T \langle J\dot{u},\, u\rangle\, dt = \frac{1}{2}\int_0^T \Big\langle \int_0^t \dot{u}(s)\, ds - \int_t^T \dot{u}(s)\, ds,\, J\dot{u}\Big\rangle\, dt + \langle \frac{u(0)+u(T)}{2},\, \int_0^T J\dot{u}(t)\, dt\rangle$$

Hence

$$\int_0^T \langle J\dot{u}, u\rangle\, dt - \langle \frac{u(0)+u(T)}{2},\, J\big(u(T)-u(0)\big)\rangle = \frac{1}{2}\int_0^T \Big\langle \int_0^t \dot{u}(s)\, ds - \int_t^T \dot{u}(s)\, ds,\, J\dot{u}\Big\rangle\, dt$$

and since J is skew-symmetric, we have

$$\int_0^T \langle J\dot{u}, u\rangle\, dt + \langle J\frac{u(0)+u(T)}{2}, u(T)-u(0)\rangle = \frac{1}{2}\int_0^T \Big\langle \int_0^t \dot{u}(s)\, ds - \int_t^T \dot{u}(s)\, ds,\, J\dot{u}\Big\rangle\, dt \tag{43}$$

Applying Hölder's inequality for the right hand side, we get

$$\Big|\int_0^T \langle J\dot{u}, u\rangle\, dt + \langle J\frac{u(0)+u(T)}{2}, u(T)-u(0)\rangle\Big| \le \frac{T}{2}\int_0^T \big|\dot{u}(t)\big|^2\, dt.$$

For part (ii), set $v(t) = u(t) - \frac{u(0)+u(T)}{2}$ and note that

$$\int_0^T \langle J\dot{u}, u\rangle\, dt + \langle J\frac{u(0)+u(T)}{2}, u(T)-u(0)\rangle = \int_0^T (J\dot{v}, v)\, dt \tag{44}$$

Since $v \in H^1_{-T}$, Hölder's inequality and Proposition 3.1 imply,

$$\begin{aligned}
\Big|\int_0^T (J\dot{v}, v)\, dt\Big| &\le \Big(\int_0^T |v|^2\, dt\Big)^{\frac{1}{2}} \Big(\int_0^T |J\dot{v}|^2\, dt\Big)^{\frac{1}{2}} \\
&\le \frac{T}{\pi}\Big(\int_0^T |\dot{v}|^2\, dt\Big)^{\frac{1}{2}} \Big(\int_0^T |J\dot{v}|^2\, dt\Big)^{\frac{1}{2}} \\
&= \frac{T}{\pi}\int_0^T |\dot{v}|^2\, dt = \frac{T}{\pi}\int_0^T |\dot{u}|^2\, dt.
\end{aligned}$$

Combining this inequality with (44) yields the claimed inequality.

Proposition 3.3 *If $H = \mathbb{R}^N$ and $X = H \times H$, then the functional $F : A_X^2 \to \mathbb{R}$ defined by*

$$F(u) = \int_0^T \langle J\dot{u}, u\rangle\, dt + \langle u(T) - u(0), J\frac{u(T)+u(0)}{2}\rangle$$

is weakly continuous.

Proof: Let u_k be a sequence in A^2_X which converges weakly to u in A^2_X. The injection A^2_X into $C([0,T];X)$ with natural norm $\| \|_\infty$ is compact, hence $u_k \to u$ strongly in $C([0,T];X)$ and specifically $u_k(T) \to u(T)$ and $u_k(0) \to u(0)$ strongly in X. Therefore

$$\lim_{k\to+\infty} \Big(u_k(T) - u_k(0), J\frac{u_k(T)+u_k(0)}{2} \Big) = \Big(u(T) - u(0), J\frac{u(T)+u(0)}{2} \Big) \tag{45}$$

Also, it is standard that $u \to \int_0^T (J\dot u, u)\,dt$ is weakly continuous (Proposition 1.2 in [9]) which together with (45) imply that F is weakly continuous.

3.1 A general variational principle for Hamiltonian systems

In this section we establish Theorem 1.2 under the assumption that H is finite dimensional ($X = \mathbb{R}^{2N}$). We start with the following proposition which assumes a stronger condition on the boundary Lagrangian.

Proposition 3.4 *Let $\varphi : [0,T] \times X \to \mathbb{R}$, such that $(t,u) \to \varphi(t,u)$ is measurable in t for each $u \in X$, and is convex and lower semi-continuous in u for a.e. $t \in [0,T]$. Let $\psi : X \to \mathbb{R} \cup \{\infty\}$ be convex and lower semi continuous and assume the following conditions:*

- *(B_1) There exists $\beta \in (0, \frac{\pi}{2T})$ and $\gamma, \alpha \in L^2(0,T;\mathbb{R}_+)$ such that $-\alpha(t) \le \varphi(t,u) \le \frac{\beta}{2}|u|^2 + \gamma(t)$ for every $u \in X$ and a.e. $t \in [0,T]$.*
- *(B_2') There exist positive constants $\alpha_1, \beta_1, \gamma_1 \in \mathbb{R}$ such that, for every $u \in X$ one has $-\alpha_1 \le \psi(u) \le \frac{\beta_1}{2}|u|^2 + \gamma_1$.*

(1) The infimum of the functional

$$\begin{aligned} J_1(u) &= \int_0^T [\varphi(t,u(t)) + \varphi^*(t, -J\dot u(t)) + \langle J\dot u(t), u(t)\rangle]\,dt \\ &\quad + \langle u(T) - u(0), J\frac{u(0)+u(T)}{2}\rangle + \psi\big(u(T)-u(0)\big) + \psi^*\big(-J\frac{u(0)+u(T)}{2}\big) \end{aligned}$$

on A^2_X is then equal to zero and is attained at a solution of

$$\begin{cases} -J\dot u(t) &= \partial\varphi\big(t,u(t)\big) \\ -J\frac{u(T)+u(0)}{2} &= \partial\psi\big(u(T)-u(0)\big). \end{cases} \tag{46}$$

(2) The infimum of the functional

$$J_2(u) = \int_0^T \big[\varphi\big(t,u(t)\big) + \varphi^*(t,-J\dot u(t)) + \langle J\dot u(t), u(t)\rangle\big]\,dt + \big(Ju(0), u(T)\big) + \psi\big(u(0)\big) + \psi^*\big(Ju(T)\big)$$

on A^2_X is also equal to zero and is attained at a solution of

$$\begin{cases} -J\dot u(t) &= \partial\varphi\big(t,u(t)\big) \\ Ju(T) &= \partial\psi\big(u(0)\big). \end{cases} \tag{47}$$

The proof requires a few preliminary lemmas, but first and anticipating that the conjugate φ^* and ψ^* may not be finite everywhere, we start by replacing φ and ψ with the perturbations such as $\varphi_\epsilon(t,u) = \frac{\epsilon}{2}\|u\|^2 + \varphi(t,u)$ and $\psi_\epsilon(u) = \frac{\epsilon}{2}\|u\|^2 + \psi(u)$. It is then clear that

$$\frac{1}{2(\beta+\epsilon)}|u|^2 - \gamma(t) \le \varphi^*_\epsilon(t,u) \le \frac{1}{2\epsilon}|u|^2 + \alpha(t), \tag{48}$$

and

$$\frac{1}{2(\beta_1+\epsilon)}|u|^2 - \gamma_1 \le \psi^*_\epsilon(u) \le \frac{1}{2\epsilon}|u|^2 + \alpha_1. \tag{49}$$

We now consider the Lagrangian $\mathcal{L}_\epsilon : A^2_X \times A^2_X \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{L}_\epsilon(v;u) &= \int_0^T [-\langle J\dot v(t), u(t)\rangle + \varphi^*_\epsilon(t,-J\dot u(t)) - \varphi^*_\epsilon(t,-J\dot v(t)) + \langle J\dot u(t), u(t)\rangle]\,dt \\ &\quad + \Big\langle u(T)-u(0), J\frac{u(T)+u(0)}{2}\Big\rangle - \Big\langle u(T)-u(0), J\frac{v(T)+v(0)}{2}\Big\rangle \\ &\quad + \psi^*_\epsilon\big(-J\frac{u(T)+u(0)}{2}\big) - \psi^*_\epsilon\big(-J\frac{v(T)+v(0)}{2}\big) \end{aligned}$$

and

$$J_1^\epsilon(u): \;=\; \int_0^T \big[\varphi_\epsilon(t,u(t)) + \varphi_\epsilon^*(t,-J\dot u(t)) + \big(J\dot u(t),u(t)\big)\big]\;dt$$
$$+\big\langle u(T)-u(0), J\frac{u(0)+u(T)}{2}\big\rangle + \psi_\epsilon\big(u(T)-u(0)\big) + \psi_\epsilon^*\big(-J\frac{u(0)+u(T)}{2}\big)$$

To simplify the notation we use C as a general positive constant.

Lemma 3.1 *For every $u\in A_X^2$, we have $J_1(u)\geq 0$ and $J_1^\epsilon(u)\geq 0$.*

Proof: By the definition of Legendre-Fenchel duality, one has

$$\varphi(t,u(t)) + \varphi^*(t,-J\dot u(t)) + \langle J\dot u(t),u(t)\rangle \geq 0 \qquad \text{for } t\in[0,T],$$

and

$$\psi\big(u(T)-u(0)\big) + \psi^*\big(-J\frac{u(0)+u(T)}{2}\big) + \langle u(T)-u(0), J\frac{u(0)+u(T)}{2}\rangle \geq 0,$$

which means $J_1(u)\geq 0$. The same applies to J_1^ϵ.

Lemma 3.2 *For every $u\in A_X^2$, we have $I_\epsilon(u)=\sup_{v\in A_X^2}\mathcal{L}_\epsilon(v,u)$.*

Proof: First recall that one can identify A_X^2 with $X\times L_X^2$ via the correspondence:

$$\begin{aligned}\big(x,f(t)\big) \;&\in\; X\times L_X^2 \longmapsto x+\frac{1}{2}\left(\int_0^t f(s)\,ds - \int_t^T f(s)\,ds\right)\in A_X^2\\ u \;&\in\; A_X^2 \longmapsto \left(\frac{u(0)+u(T)}{2},\dot u(t)\right)\in X\times L_X^2\end{aligned}$$

Thus, for every $u\in A_X^2$, we can write

$$\begin{aligned}\sup_{v\in A_X^2}\mathcal{L}_\epsilon(v;u) &= \sup_{v\in X\times L_X^2}\mathcal{L}_\epsilon(v,u)\\
&= \sup_{f\in L^2(0,T;X)}\sup_{x\in X}\Big\{\int_0^T \big[\langle -Jf(t),u(t)\rangle + \varphi_\epsilon^*(t,-J\dot u(t)) - \varphi_\epsilon^*(t,-Jf(t)) + (J\dot u(t),u(t))\big]\,dt\Big\}\\
&\quad +\big\langle u(T)-u(0), J\frac{u(T)+u(0)}{2}\big\rangle - \big\langle u(T)-u(0), Jx\big\rangle + \psi_\epsilon^*\big(-J\frac{u(T)+u(0)}{2}\big) - \psi_\epsilon^*(-Jx)\\
&= \sup_{f\in L^2(0,T;X)}\Big\{\int_0^T \big[\langle -Jf(t),u(t)\rangle + \varphi_\epsilon^*(t,-J\dot u(t)) - \varphi_\epsilon^*(t,-Jf(t)) + (J\dot u(t),u(t))\big]\,dt\Big\}\\
&\quad + \sup_{x\in X}\Big\{\big\langle u(T)-u(0), J\frac{u(T)+u(0)}{2}\big\rangle - \big\langle u(T)-u(0), Jx\big\rangle\\
&\quad +\psi_\epsilon^*\big(-J\frac{u(T)+u(0)}{2}\big) - \psi_\epsilon^*(-Jx)\Big\}\\
&= \int_0^T [\varphi_\epsilon(t,u(t)) + \varphi_\epsilon^*(t,-J\dot u(t)) + (J\dot u(t),u(t))]\;dt\\
&\quad +\big\langle u(T)-u(0), J\frac{u(T)+u(0)}{2}\big\rangle + \psi_\epsilon\big(u(T)-u(0)\big) + \psi_\epsilon^*\big(-J\frac{u(T)+u(0)}{2}\big)\\
&= J_1^\epsilon(u).\end{aligned}$$

Lemma 3.3 *Under the assumptions (B_1) and (B_2'), we have for each $0<\epsilon<\frac{1}{2}(\frac{\pi}{T}-2\beta)$ the following coercivity condition*

$$\mathcal{L}_\epsilon(0,u)\to+\infty \quad \text{when} \quad \|u\|_{A_X^2}\to+\infty. \tag{50}$$

Proof: From (48) and (49) and since $\int_0^T \varphi^*(t,0)\,dt$ and $\psi^*(0)$ are finite, we get

$$\begin{aligned}\mathcal{L}_\epsilon(0,u) \quad &\geq \quad \frac{1}{2(\beta+\epsilon)}\int_0^T |\dot u(t)|^2\,dt + \int_0^T \langle J\dot u(t), u(t)\rangle\,dt + \langle J\frac{u(0)+u(T)}{2}, u(T)-u(0)\rangle \\ &\quad + \frac{1}{2(\beta_1+\epsilon)}\left|\frac{u(0)+u(T)}{2}\right|^2 + C,\end{aligned}$$

where C is a constant. From part (ii) of Proposition 3.2, we have

$$\left|\int_0^T \langle J\dot u(t), u(t)\rangle\,dt + \langle u(T)-u(0), J\frac{u(T)+u(0)}{2}\rangle\right| \leq \frac{T}{\pi}\int_0^T |\dot u(t)|^2\,dt$$

Hence, modulo a constant, we obtain

$$\mathcal{L}_\epsilon(0,u) \geq \left(\frac{1}{2(\beta+\epsilon)} - \frac{T}{\pi}\right)\int_0^T |\dot u(t)|^2\,dt + \frac{1}{2(\beta_1+\epsilon)}\left|\frac{u(0)+u(T)}{2}\right|^2.$$

Since $0<\epsilon<\frac{1}{2}(\frac{\pi}{T}-2\beta)$, it follows that $\frac{1}{2(\beta+\epsilon)}-\frac{T}{\pi}>0$ and $\mathcal{L}_\epsilon(0,u)\to+\infty$ as $\|u\|_{A^2_X}\to+\infty$. □
Proposition 3.4 is now a consequence of the following Ky-Fan type min-max theorem which is essentially due to Brezis-Nirenberg-Stampachia (see [2]).

Lemma 3.4 *Let Y be a a reflexive Banach space and let $\mathcal{L}(x,y)$ be a real valued function on $Y\times Y$ that satisfies the following conditions:*

(1) $\mathcal{L}(x,x)\leq 0$ for every $x\in Y$.

(2) For each $x\in Y$, the function $y\to\mathcal{L}(x,y)$ is concave.

(3) For each $y\in Y$, the function $x\to\mathcal{L}(x,y)$ is weakly lower semi-continuous.

(4) The set $Y_0=\{x\in Y;\mathcal{L}(x,0)\leq 0\}$ is bounded in Y.

Then there exists $x_0\in Y$ such that $\sup_{y\in Y}\mathcal{L}(x_0,y)\leq 0$.

Proof of Proposition 3.4: Let $0<\delta<\frac{1}{2}(\frac{\pi}{T}-2\beta)$ and $0<\epsilon<\delta$. It is easy to see that the $\mathcal{L}_\epsilon: X\times X\to\mathbb{R}$ satisfies all the hypothesis of Lemma 3.4. It follows from (48) and (49) that $\mathcal{L}_\epsilon$ is finitely valued on $X\times X$ and that for each $u\in X\times X$, $\mathcal{L}_\epsilon(u,u)=0$. Lemma 3.3 gives that the set $Y=\{u\in X,\mathcal{L}_\epsilon(0,u)\leq 0\}$ is bounded in X. Moreover, for every $u\in X$, the function $v\to\mathcal{L}_\epsilon(v,u)$ is concave and for every $v\in X$, $u\to\mathcal{L}_\epsilon(u,v)$ is weakly lower semi-continuous by Proposition 3.3. It follows that there exists $u_\epsilon\in X$ such that $I_\epsilon(u_\epsilon)\leq\sup_{v\in A^2_X}\mathcal{L}_\epsilon(v,u_\epsilon)\leq 0$.

In view of Lemma 3.1, we then have $I_\epsilon(u_\epsilon)=0$ which yields:

$$\begin{aligned}I_\epsilon(u_\epsilon) \quad &= \quad \int_0^T [\varphi_\epsilon(t,u_\epsilon(t)) + \varphi^*_\epsilon(t,-J\dot u_\epsilon(t)) + \langle u_\epsilon(t), J\dot u_\epsilon(t)\rangle]\,dt \\ &\qquad + \psi_\epsilon\big(u_\epsilon(T)-u_\epsilon(0)\big) + \psi^*_\epsilon\big(-J\frac{u_\epsilon(0)+u_\epsilon(T)}{2}\big) \\ &\qquad + \langle u_\epsilon(T)-u_\epsilon(0), J\frac{u_\epsilon(0)+u_\epsilon(T)}{2}\rangle \\ &= \quad 0. \end{aligned} \tag{51}$$

We shall show that u_ϵ is bounded in X. From Proposition 3.2, we have

$$\left|\int_0^T (J\dot u_\epsilon(t), u_\epsilon(t))\,dt + \langle u_\epsilon(T)-u_\epsilon(0), J\frac{u_\epsilon(T)+u_\epsilon(0)}{2}\rangle\right| \leq \frac{T}{\pi}\int_0^T |\dot u_\epsilon(t)|^2\,dt$$

which together with (51), yield

$$\int_0^T [\varphi_\epsilon(t,u_\epsilon(t)) + \varphi^*_\epsilon(t,-J\dot u_\epsilon(t))]\,dt - \frac{T}{\pi}\int_0^T |\dot u_\epsilon(t)|^2\,dt + \psi_\epsilon\big(u_\epsilon(T)-u_\epsilon(0)\big) + \psi^*_\epsilon\big(-J\frac{u_\epsilon(0)+u_\epsilon(T)}{2}\big) \leq 0.$$

This inequality together with the facts that φ_ϵ and ψ_ϵ are bounded from below and φ^*_ϵ and ψ^*_ϵ satisfy inequalities (48) and (49) respectively, guarantee the existence of a constant $C>0$ independent of ϵ such that

$$\begin{aligned}\Big(\frac{1}{2(\beta+\delta)}-\frac{T}{\pi}\Big)\int_0^T |\dot u_\epsilon(t)|^2\,dt+\frac{1}{2(\beta_1+\delta)}\Big|\frac{u_\epsilon(0)+u_\epsilon(T)}{2}\Big|^2 &\le \\ \Big(\frac{1}{2(\beta+\epsilon)}-\frac{T}{\pi}\Big)\int_0^T |\dot u_\epsilon(t)|^2\,dt+\frac{1}{2(\beta_1+\epsilon)}\Big|\frac{u_\epsilon(0)+u_\epsilon(T)}{2}\Big|^2 &\le C,\end{aligned}$$

which means $(u_\epsilon)_\epsilon$ is bounded in A^2_X and so, up to a subsequence , there exists a $\bar u\in A^2_X$ such that $u_\epsilon\rightharpoonup \bar u$ in A^2_X. It is easily seen that

$$\int_0^T \varphi^*_\epsilon(t,\dot u_\epsilon(t))\,dt := \inf_{v\in L^2(0,T;X)}\int_0^T\Big[\varphi^*(t,v(t))+\frac{|\dot u_\epsilon(t)-v(t)|^2}{2\epsilon}\Big]\,dt$$

and since φ^* is convex and lower semi continuous, there exists $v_\epsilon\in L^2(0,T;X)$ such that this infimum attains at v_ϵ, i.e.

$$\int_0^T \varphi^*_\epsilon(t,\dot u_\epsilon(t))\,dt = \int_0^T\Big[\varphi^*(t,v_\epsilon(t))+\frac{|\dot u_{\epsilon(t)}-v_\epsilon(t)|^2}{2\epsilon}\Big]\,dt.$$

It follows from the above and the boundedness of $(u_\epsilon)_\epsilon$ in A^2_X, that there exists $C>0$ independent of ϵ such that

$$\int_0^T \varphi^*_\epsilon(t,\dot u_\epsilon(t))\,dt = \int_0^T\Big[\varphi^*(t,v_\epsilon(t))+\frac{|\dot u_\epsilon(t)-v_\epsilon(t)|^2}{2\epsilon}\Big]\,dt<C.$$

Since φ^* is bounded from below, we have $\int_0^T|\dot u_\epsilon(t)-v_\epsilon(t)|^2dt<C\epsilon$ which means $v_\epsilon\rightharpoonup\dot{\bar u}$ in $L^2(0,T;X)$. Hence

$$\begin{aligned}\int_0^T\varphi^*(t,\dot{\bar u}(t))\,dt &\le \liminf_{\epsilon\to0}\int_0^T\varphi^*(t,v_\epsilon(t))dt\\ &\le \liminf_{\epsilon\to0}\int_0^T\Big[\varphi^*(t,v_\epsilon(t))+\frac{|\dot u_\epsilon(t)-v_\epsilon(t)|^2}{2\epsilon}\Big]\,dt\\ &= \liminf_{\epsilon\to0}\int_0^T\varphi^*_\epsilon(t,\dot u_\epsilon(t))dt. \qquad (52)\end{aligned}$$

Also,

$$\begin{aligned}\int_0^T\varphi(t,\bar u(t))\,dt &\le \liminf_{\epsilon\to0}\int_0^T\varphi(t,u_\epsilon(t))\,dt\\ &\le \liminf_{\epsilon\to0}\int_0^T\Big[\varphi(t,u_\epsilon(t))+\frac{\epsilon}{2}|u_\epsilon(t)|^2\Big]\,dt\\ &= \liminf_{\epsilon\to0}\int_0^T\varphi_\epsilon(t,u_\epsilon(t))dt. \qquad (53)\end{aligned}$$

It follows from (52) and (53) that,

$$\int_0^T\big[\varphi(t,\bar u(t))+\varphi^*(t,-J\dot{\bar u}(t))\big]\,dt\le\liminf_{\epsilon\to0}\int_0^T\big[\varphi_\epsilon(t,u_\epsilon(t))+\varphi^*_\epsilon(t,-J\dot u_\epsilon(t))\big]\,dt. \qquad (54)$$

By the same argument we arrive at,

$$\psi\big(\bar u(T)-\bar u(0)\big)+\psi^*\big(-J\frac{\bar u(0)+\bar u(T)}{2}\big)\le\liminf_{\epsilon\to0}\Big\{\psi_\epsilon\big(u_\epsilon(T)-u_\epsilon(0)\big)+\psi^*_\epsilon\big(-J\frac{u_\epsilon(0)+u_\epsilon(T)}{2}\big)\Big\}$$

Also, from Proposition 3.3, we have

$$\begin{aligned}\lim_{\epsilon\to0}\int_0^T\langle u_\epsilon(t),J\dot u_\epsilon(t)\rangle\,dt &+ \langle u_\epsilon(T)-u_\epsilon(0),J\frac{u_\epsilon(0)+u_\epsilon(T)}{2}\rangle\\ &= \int_0^T\langle\bar u(t),J\dot{\bar u}(t)\rangle dt+\langle\bar u(T)-\bar u(0),J\frac{\bar u(0)+\bar u(T)}{2}\rangle. \qquad (55)\end{aligned}$$

Combining the above yields

$$\begin{aligned}
I(\bar u) &= \int_0^T \big[\varphi(t,\bar u(t))+\varphi^*(t,-J\dot{\bar u}(t))+(\bar u(t),J\dot{\bar u}(t))\big]\,dt\\
&+ \psi\big(\bar u(T)-\bar u(0)\big)+\psi^*\big(-J\frac{\bar u(0)+\bar u(T)}{2}\big)+\langle \bar u(T)-\bar u(0),J\frac{\bar u(0)+\bar u(T)}{2}\rangle\\
&\le \liminf_{\epsilon\to 0}\Big\{\int_0^T [\varphi_\epsilon(t,u_\epsilon(t))+\varphi_\epsilon^*(t,-J\dot u_\epsilon(t))+(u_\epsilon(t),J\dot u_\epsilon(t))]\,dt\\
&+ \psi_\epsilon\big(u_\epsilon(T)-u_\epsilon(0)\big)+\psi_\epsilon^*\big(-J\frac{u_\epsilon(0)+u_\epsilon(T)}{2}\big)+\langle u_\epsilon(T)-u_\epsilon(0),J\frac{u_\epsilon(0)+u_\epsilon(T)}{2}\rangle\Big\}\\
&= \liminf_{\epsilon\to 0} I_\epsilon(u_\epsilon)=0.
\end{aligned}$$

On the other hand Lemma 3.1 implies that $I(\bar u)\ge 0$, which means the latter is zero, i.e.

$$\begin{aligned}
I(\bar u) &= \int_0^T \big[\varphi(t,\bar u)+\varphi^*(t,-J\dot{\bar u})+(\bar u,J\dot{\bar u})\big]\,dt\\
&+ \psi\big(\bar u(T)-\bar u(0)\big)+\psi^*\big(-J\frac{\bar u(0)+\bar u(T)}{2}\big)+\langle \bar u(T)-\bar u(0),J\frac{\bar u(0)+\bar u(T)}{2}\rangle=0.
\end{aligned}$$

The result now follows from the following identities and from the limiting case in Legendre-Fenchel duality.

$$\varphi(t,\bar u(t))+\varphi^*(t,-J\dot{\bar u}(t))+(\bar u(t),J\dot{\bar u}(t))=0$$

$$\psi\big(\bar u(T)-\bar u(0)\big)+\psi^*\big(-J\frac{\bar u(0)+\bar u(T)}{2}\big)+\langle \bar u(T)-\bar u(0),J\frac{\bar u(0)+\bar u(T)}{2}\rangle=0.$$

□

We shall now use Proposition 3.4 to prove Theorem 1.2. For that we shall $\lambda-$regularize the convex functional ψ, then use assumption B_2 of Theorem 1.2 to derive uniform bounds and ensure convergence in A^2_X when λ approaches to 0. First recall that if $\psi_\lambda(x)=\inf_{y\in X}\Big\{\psi(y)+\frac{\|x-y\|^2_X}{2\lambda}\Big\}$ then its conjugate ψ^*_λ is equal to $\psi^*(x)+\frac{\lambda|x|^2}{2}$, which means that if $G(x,p)$ is the anti-selfdual Lagrangian $G(x,p)=\psi(x)+\psi^*(-p)$, then its λ-regularization is nothing but $G_\lambda(x,p)=\psi_\lambda(x)+\psi^*_\lambda(-p)$.

Proof of Theorem 1.2, Part (1): The functional ψ_λ satisfies the condition (B_2') of Proposition 3.4, hence for each $\lambda>0$ there exists a $u_\lambda\in A^2_X$, such that

$$\begin{aligned}
I_\lambda(u_\lambda) &:= \int_0^T [\varphi(t,u_\lambda(t))+\varphi^*(t,-J\dot u_\lambda(t))+\langle J\dot u_\lambda(t),u_\lambda(t)\rangle]\,dt\\
&\quad +\langle u_\lambda(T)-u_\lambda(0),J\frac{u_\lambda(T)+u_\lambda(0)}{2}\rangle+\psi_\lambda\big(u_\lambda(T)-u_\lambda(0)\big)+\psi^*_\lambda\big(-J\frac{u_\lambda(T)+u_\lambda(0)}{2}\big)\\
&= 0. \qquad (56)
\end{aligned}$$

We shall show u_λ is bounded in A^2_X. From Proposition 3.2 we obtain

$$\Big|\int_0^T\langle J\dot u_\lambda,u_\lambda\rangle\,dt+\langle u_\lambda(T)-u_\lambda(0),J\frac{u_\lambda(T)+u_\lambda(0)}{2}\rangle\Big|\le\frac{T}{\pi}\int_0^T|\dot u_\lambda(t)|^2\,dt$$

which together with (48) and (56) imply

$$\psi_\lambda\big(u_\lambda(T)-u_\lambda(0)\big)+\psi^*_\lambda\big(-J\frac{u_\lambda(T)+u_\lambda(0)}{2}\big)+\int_0^T\varphi(t,u_\lambda(t))\,dt+\Big(\frac{1}{2\beta}-\frac{T}{\pi}\Big)\int_0^T|\dot u_\lambda(t)|^2\,dt\le 0. \quad (57)$$

Since ψ is bounded from below so is ψ_λ. Also, $0\in$ Dom(ψ) which means ψ^* and consequently ψ^*_λ is bounded from below. Therefore it follows from (57) that:

$$\psi_\lambda\big(u_\lambda(T)-u_\lambda(0)\big)+\psi^*_\lambda\big(-J\frac{u_\lambda(T)+u_\lambda(0)}{2}\big)\le C, \qquad (58)$$

and

$$\int_0^T \varphi(t,u_\lambda)\,dt + \left(\frac{1}{2\alpha} - \frac{T}{2}\right)\int_0^T |\dot u_\lambda(t)|^2 \le C, \tag{59}$$

where $C>0$ is a positive constant. It follows from the assumption $(B_1),(B_2)$ and (59) that $|u_\lambda(t)|$ and $\int_0^T |\dot u_\lambda|^2\,dt$ are bounded. Consequently u_λ is bounded in A^2_X and so, up to a subsequence, $u_\lambda \rightharpoonup \bar u$ in A^2_X.

It follows from (58) and Lemma 2.2 that

$$\psi\big(\bar u(T)-\bar u(0)\big) + \psi^*\big(-J\frac{\bar u(T)+\bar u(0)}{2}\big) \le \liminf_\lambda \psi_\lambda\big(u_\lambda(T)-u_\lambda(0)\big) + \psi^*_\lambda\big(-J\frac{u_\lambda(T)+\bar u_\lambda(0)}{2}\big). \tag{60}$$

Also, from Proposition 3.3, we have

$$\begin{aligned}\inf_{\lambda\to 0}\int_0^T \langle u_\lambda(t), J\dot u_\lambda(t)\rangle\,dt \quad &+ \quad \langle u_\lambda(T)-u_\lambda(0), J\frac{u_\lambda(0)+u_\lambda(T)}{2}\rangle \\ &= \int_0^T \langle \bar u(t), J\dot{\bar u}(t)\rangle\,dt + \langle \bar u(T)-\bar u(0), J\frac{\bar u(0)+\bar u(T)}{2}\rangle. \end{aligned}\tag{61}$$

Now, taking into account (60) and (61), by letting $\lambda\to 0$ in (56) we obtain,

$$\begin{aligned} I(\bar u) &= \int_0^T \big[\varphi(t,\bar u(t)) + \varphi^*\big(t,-J\dot{\bar u}(t)\big) + \langle J\dot{\bar u}(t),\bar u(t)\rangle\big]\,dt \\ &\quad + \langle \bar u(T)-\bar u(0), J\frac{\bar u(T)+\bar u(0)}{2}\rangle + \psi\big(\bar u(T)-\bar u(0)\big) + \psi^*\big(-J\frac{\bar u(T)+\bar u(0)}{2}\big) \\ &\le \liminf_{\lambda\to 0}\Big\{\int_0^T [\varphi(t,u_\lambda(t)) + \varphi^*(t,-J\dot u_\lambda(t)) + \langle J\dot u_\lambda(t),u_\lambda(t)\rangle]\,dt \\ &\quad + \langle u_\lambda(T)-u_\lambda(0), J\frac{u_\lambda(T)+u_\lambda(0)}{2}\rangle + \psi_\lambda\big(u_\lambda(T)-u_\lambda(0)\big) + \psi^*_\lambda\big(-J\frac{u_\lambda(T)+u_\lambda(0)}{2}\big)\Big\} \\ &= \liminf_{\lambda\to 0} I_\lambda(u_\lambda) = 0. \end{aligned}$$

From Lemma 3.1, $I(\bar u)\ge 0$, which means the latter is zero. The result follows from the following identities and from the limiting case in Legendre-Fenchel duality

$$\varphi(t,\bar u) + \varphi^*(t,-J\dot{\bar u}) + \langle \bar u, J\dot{\bar u}\rangle = 0.$$

$$\psi\big(\bar u(T)-\bar u(0)\big) + \psi^*\big(-J\frac{\bar u(0)+\bar u(T)}{2}\big) + \langle \bar u(T)-\bar u(0), J\frac{\bar u(0)+\bar u(T)}{2}\rangle = 0.$$

Proof of Part (2): Note first that $\langle J\frac{u(0)+u(T)}{2}, u(T)-u(0)\rangle = \langle Ju(0), u(T)\rangle$. The corresponding Lagrangian $L_\epsilon : X\times X \to \mathbb{R}$ is defined as follows

$$\begin{aligned} L_\epsilon(v,u) &= \int_0^T [(-J\dot v(t), u(t)) + \varphi^*_\epsilon(t,-J\dot u(t)) - \varphi^*_\epsilon(t,-J\dot v(t)) + \langle J\dot u(t),u(t)\rangle]\,dt \\ &\quad + \langle Ju(0),u(T)\rangle - \langle Ju(0),v(T)\rangle + \psi^*_\epsilon\big(Ju(T)\big) - \psi^*_\epsilon\big(Jv(T)\big), \end{aligned}$$

The rest of the proof is quite similar to Part (1) and is left to the interested reader.

3.2 Applications

As mentioned in the introduction, one can choose the boundary Lagrangian ψ appropriately to solve Hamiltonian systems of the form

$$\begin{cases} -J\dot u(t) \in \partial\varphi(t,u(t)) \\ u(0)=u_0, \text{ or } u(T)-u(0)\in K, \text{ or } u(T)=-u(0) \text{ or } u(T)=Ju(0). \end{cases}$$

One can also use the method to solve second order systems with convex potential and with prescribed nonlinear boundary conditions such as:

$$\begin{cases} -\ddot q(t) &= \partial\varphi\big(t,q(t)\big) \\ -\frac{q(0)+q(T)}{2} &= \partial\psi_1\big(\dot q(T)-\dot q(0)\big), \\ \frac{\dot q(0)+\dot q(T)}{2} &= \partial\psi_2\big(q(T)-q(0)\big) \end{cases}\tag{62}$$

and

$$\begin{cases} \ddot{q}(t) &= \partial\varphi\big(t,q(t)\big) \\ -q(T) &= \partial\psi_1\big(\dot{q}(0)\big), \\ \dot{q}(T) &= \partial\psi_2\big(q(0)\big) \end{cases} \tag{63}$$

where ψ_1 and ψ_2 are convex and lower semi continuous. One can deduce the following

Corollary 3.5 *Let $\varphi : [0,T] \times H \to \mathbb{R}$ be such that $(t,q) \to \varphi(t,q)$ is measurable in t for each $q \in H$, convex and lower semi-continuous in q for a.e. $t \in [0,T]$, and let $\psi_i : H \to \mathbb{R} \cup \{\infty\}$, $i = 1,2$ be convex and lower semi continuous on H. Assume that the following conditions:*

A_1**:** *There exists $\beta \in (0, \frac{\pi}{2T})$ and $\gamma, \alpha \in L^2(0,T;\mathbb{R}_+)$ such that $-\alpha(t) \le \varphi(t,q) \le \frac{\beta^2}{2}|q|^2 + \gamma(t)$ for every $q \in H$ and a.e. $t \in [0,T]$.*

A_2**:** $\int_0^T \varphi(t,q)\,dt \to +\infty$ *as* $|q| \to +\infty$.

A_3**:** *ψ_1 and ψ_2 are bounded from below and $0 \in \mathrm{Dom}(\psi_i)$ for $i = 1,2$.*

Then equations (62) and (63) have at least one solution in A_H^2.

Proof: Define $\Psi : H \times H \to \mathbb{R} \cup \{\infty\}$ by $\Psi(p,q) := \psi_1(p) + \psi_2(q)$ and $\Phi : [0,T] \times H \times H \to \mathbb{R}$ by $\Phi(t,u) := \frac{\beta}{2}|p|^2 + \frac{1}{\beta}\varphi\big(t,q(t)\big)$ where $u = (p,q)$. It is easily seen that Φ is convex and lower semi continuous in u and that

$$-\alpha(t) \le \Phi(t,u) \le \frac{\beta}{2}|u|^2 + \frac{\gamma(t)}{\beta} \text{ and } \int_0^T \Phi(t,u)\,dt \to +\infty \quad \text{as} \quad |u| \to +\infty.$$

Also, from A_3, the function Ψ is bounded from below and $0 \in \mathrm{Dom}(\Psi)$. By Theorem 1.2, the infimum of the functional

$$\begin{aligned} I(u) : &= \int_0^T [\Phi(t,u(t)) + \Phi^*(t, -J\dot{u}(t)) + \langle J\dot{u}(t), u(t)\rangle]\,dt \\ &\quad + \langle u(T) - u(0), J\frac{u(0)+u(T)}{2}\rangle + \Psi\big(u(T) - u(0)\big) + \Psi^*\big(-J\frac{u(0)+u(T)}{2}\big), \end{aligned}$$

on A_X^2 is zero and is attained at a solution of

$$\begin{cases} -J\dot{u}(t) \in \partial\Phi(t,u(t)), \\ -J\frac{u(T)+u(0)}{2} = \partial\Psi(u(T) - u(0)). \end{cases}$$

Now if we rewrite this problem for $u = (p,q)$, we get

$$\begin{aligned} -\dot{p}(t) &= \frac{1}{\beta}\partial\varphi\big(t,q(t)\big), \\ \dot{q}(t) &= \beta p(t), \\ -\frac{q(T)+q(0)}{2} &= \partial\psi\big(p(T) - p(0)\big), \\ \frac{p(T)+p(0)}{2} &= \partial\psi\big(q(T) - q(0)\big), \end{aligned}$$

and hence $q \in A_H^2$ is a solution of (61).

As in the case of Hamiltonian systems, one can then solve variationally the differential equation $-\ddot{q}(t) = \partial\varphi\big(t,q(t)\big)$ with any one of the following boundary conditions:

(i) Periodic: $\dot{q}(T) = \dot{q}(0)$ and $q(T) = q(0)$.

(ii) Antiperiodic: $\dot{q}(T) = -\dot{q}(0)$ and $q(T) = -q(0)$.

(iii) Initial value condition: $q(0) = q_0$ and $\dot{q}(0) = q_1$ for given $q_0, q_1 \in H$.

References

[1] H. Brezis, I. Ekeland, *Un principe variationnel associé à certaines equations paraboliques. Le cas independant du temps*, C.R. Acad. Sci. Paris Sér. A **282** (1976), 971–974.

[2] H. Brezis, L. Nirenberg, G. Stampachia, *A remark on Ky Fan's Minimax Principle*, Bollettino U. M. I (1972), 293-300

[3] N. Ghoussoub, *Anti-selfdual Lagrangians: Variational resolutions of non self-adjoint equations and dissipative evolutions*, Submitted (2005).

[4] N. Ghoussoub, *Anti-selfdual Hamiltonians: Variational resolution for Navier-Stokes equations and other nonlinear evolutions*, Submitted (2005)

[5] N. Ghoussoub, *A class of selfdual partial differential equations and its variational principles*, In preparation (2005)

[6] N. Ghoussoub, A. Moameni, *On the existence of Hamiltonian paths connecting Lagrangian submanifolds*, Submitted (2005)

[7] N. Ghoussoub, L. Tzou. *A variational principle for gradient flows*, Math. Annalen, Vol 30, 3 (2004) p. 519-549.

[8] N. Ghoussoub, L. Tzou. *Iterations of anti-selfdual Lagrangians and applications to Hamiltonian systems and multiparameter gradient flows*, Submitted (2005).

[9] J. Mawhin, M. Willem: *Critical point theory and Hamiltonian systems.* Applied Mathematical Sciences, **74**, Springer Verlag (1989).